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THE JOHNS HOPKINS UNIVERSITY

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# UNIFORMLY BOUNDED REPRESENTATIONS AND HARMONIC ANALYSIS OF THE $2 \times 2$ REAL UNIMODULAR GROUP.\*<sup>1</sup>

By R. A. KUNZE and E. M. STEIN.

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**Introduction.** This paper deals with a study of the real  $2 \times 2$  unimodular group. Our study of this particular group is motivated by two factors. First, this group has an intrinsic interest, especially in view of its connection with several branches of Analysis. Secondly, the  $2 \times 2$  real unimodular group affords an illuminating example for the study of other groups.

We construct a family of uniformly bounded representations of the group, and consider its implication with regard to the Fourier analysis of the group. These representations are constructed with the following properties. They all act on a *fixed* Hilbert space  $\mathcal{H}$ ; they are determined by a complex parameter  $s$ ,  $0 < R(s) < 1$ , and depend analytically on the parameter  $s$ ; finally, when  $R(s) = \frac{1}{2}$ , these representations are, up to unitary equivalence, the continuous principal series.

The above properties, in particular the analyticity, together with certain convexity arguments applied to operator valued functions yield the following:

1) The "Fourier-Laplace" transform of a function  $f$  in  $L_1(G)$  exists as an operator-valued function  $\mathcal{F}$ , whose values  $\mathcal{F}(s)$  act on  $\mathcal{H}$ , and which is analytic in  $s$ ,  $0 < R(s) < 1$ .

2) When  $f \in L_p(G)$ ,  $1 \leq p < 2$ , the Fourier-Laplace transform  $\mathcal{F}$  can still be defined, and is an operator valued function analytic in the strip,

$$1 - 1/p < R(s) < 1/p.$$

3) A detailed analysis of the proofs of the above reveals the remarkable fact: If  $f \in L_p(G)$ ,  $1 \leq p < 2$ , the Fourier-Laplace transform  $\mathcal{F}$  of  $f$  is uniformly bounded in the operator norm along the line  $R(s) = \frac{1}{2}$ .

In conjunction with an analysis of the discrete series of representations, 3) implies the following significant fact concerning harmonic analysis on the group: Let  $k$  be a function in  $L_p(G)$ ,  $1 \leq p < 2$ . In contrast with the (non-compact) abelian situation, the transformation

$$f \rightarrow f * k,$$

of convolution by  $k$ , is a bounded operator on  $L_2(G)$ .

We shall now discuss certain of these facts in greater detail. The representations we consider arise as follows. Let

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad ad - bc = 1$$

be an element of the group. We then consider, for each complex  $s$ , the multiplier representations,<sup>2</sup>

<sup>2</sup> These representations may be put in the form originally obtained by Bargmann [1] by means of the transformation  $\sigma = \tan(\theta/2)$ .

$$(1.1) \quad f(x) \rightarrow |bx + d|^{2s-2} f((ax + c)/(bx + d))$$

$$(1.2) \quad f(x) \rightarrow \operatorname{sgn}(bx + d) |bx + d|^{2s-2} f((ax + c)/(bx + d)).$$

The two continuous principal series are obtained from these by setting  $s = \frac{1}{2} + it$  and restricting the functions  $f$  to lie in  $L_2(-\infty, \infty)$ .

We are led to the construction of the uniformly bounded representations described above by the following considerations. In the group we distinguish a particular subgroup, namely, the subgroup of lower triangular matrices of the form

$$g = \begin{bmatrix} a^{-1} & 0 \\ c & a \end{bmatrix}, \quad a \neq 0.$$

It may be shown that when the representations of either of the principal series are restricted to this subgroup then they are all unitarily equivalent. This raises a natural problem. Can one find a Hilbert space  $\mathcal{H}$  and representations  $U^+(\cdot, \frac{1}{2} + it)$ ,  $U^-(\cdot, \frac{1}{2} + it)$  unitarily equivalent to (1.1), (1.2) (for  $s = \frac{1}{2} + it$ ) such that  $U^\pm(\cdot, \frac{1}{2} + it)$  when restricted to the lower triangular subgroup are independent of  $t$ ? The answer is in the affirmative; furthermore, the uniformly bounded representations  $U^\pm(\cdot, s)$ ,  $0 < R(s) < 1$ , which we construct, are characterized as the analytic continuations of the representations  $U^\pm(\cdot, \frac{1}{2} + it)$ . It should be added that the representations  $U^+(\cdot, \sigma)$ ,  $0 < \sigma < 1$ , are unitarily equivalent to the complementary series.

In solving the above problem and in the actual construction of these representations, it is natural to consider the induced action of (1.1), (1.2) on the Fourier transforms  $F$  of the functions  $f$ . The considerations here are rather involved but have an intrinsic interest. For the analysis reveals connections with both the so called "Hilbert transform" and the notion of "fractional integration."

It is clear that the multiplier representations (1.1) or (1.2) afford (at least formally) an analytic continuation of (1.1) or (1.2) when  $s = \frac{1}{2} + it$ . The problem, however, is how to make this precise, i. e., the problem of finding an underlying Hilbert space on which these representations act and depend analytically on  $s$ . Ehrenpreis and Mautner have also dealt with the problem of extending the representations (1.1) to values of  $s \neq \frac{1}{2} + it$ . Their result concerning the analyticity of the Fourier transform of an  $L_1$  spherical function on  $G$  was one of the motivating facts in our work. In addition, it was brought to our attention by Ehrenpreis that a similar result might hold for  $L_p$ . However, there are significant differences between their results and ours. In [5] they construct uniformly bounded representations arising from (1.1) when  $s \neq \frac{1}{2} + it$ ; nevertheless, these representations act on different

Hilbert spaces depending on  $s$ . In [6], and [7] they consider, at least implicitly, representations which act on a fixed Hilbert space; but in this case the representations are not uniformly bounded when  $s \neq \frac{1}{2} + it$ .

The preceding considerations, in particular 3) above, lead to characterizations of the representations of the group. This may best be understood in the following context. As a result of the work of Bargmann [1], Godement [8], and Harish-Chandra [14], attention has been focused on a particular class of representations, those which are "square integrable" in the following sense; a representation  $g \rightarrow U_g$  on a Hilbert space  $\mathcal{H}$  is of this type if the function

$$\phi = \phi(g) = (U_g \xi, \eta)$$

is in  $L_2(G)$  for every  $\xi, \eta$  in  $\mathcal{H}$ . The square integrable irreducible unitary representations of the  $2 \times 2$  real unimodular group are essentially the representations of the discrete series. We are able to give a similar characterization of the representations of the continuous principal series. An irreducible unitary representation  $g \rightarrow U_g$  is equivalent to one of the latter if and only if

$$(U_g \xi, \eta) \in L_q(G)$$

holds for all  $\xi, \eta \in \mathcal{H}$  and all  $q > 2$ , but not for  $q = 2$ . An analogous characterization holds for the representations of the complementary series. (See Theorem 10 and its corollary, in § 11.)

One of the main ideas motivating this paper was the desire to extend the classical Hausdorff-Young theorem to the group. We recall the form of the Hausdorff-Young theorem for Fourier transforms, as given by Titchmarsh. Let  $f \in L_p$ ,  $1 \leq p \leq 2$ , and let

$$F(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-ixy} f(y) dy.$$

Then

$$\|F\|_q \leq (2\pi)^{\frac{1}{2}-1/p} \|f\|_p,$$

where  $1/p + 1/q = 1$ .

The most convenient method for proving this theorem is by using a convexity principle for linear transformations introduced by M. Riesz. This principle allows one to "interpolate" between various bounds of linear transformations. For a general discussion of this method of proof we refer the reader to [3]. An extension of the above theorem to locally compact abelian groups, via the Riesz convexity principle, is given in Weil's book [23]. An abstract generalization of this theorem to arbitrary locally compact unimodular groups has been given by one of us, [16]. This general theorem was proved

by what amounts to an extension of the convexity principle to linear transformations between operator valued functions.

Due to the analytic structure of the family of uniformly bounded representations of  $G$ , it is possible to prove a version of the Hausdorff-Young theorem which is much stronger than its classical analogue (see Theorem 7 in § 8). The proof of Theorem 7 necessitates yet another extension of the Riesz convexity principle—from the case of a single fixed linear transformation to a family of transformations depending analytically on a parameter.<sup>3</sup>

It seems quite likely that many of the results described above hold not only for this group but for certain other groups as well (e.g. the complex classical groups). We hope to return to this matter at a later time.

We now proceed to describe the organization of this paper.

In Chapter I, which consists of §§ 2, 3, and 4, we consider operator valued functions and we prove the basic convexity (interpolation) theorems. §§ 2 and 3 are quite general in nature. However, in § 4 the subject matter is tailored to fit the situation which arises in the  $2 \times 2$  real unimodular group.

Chapter II concerns itself with the actual construction of our family of representations. In § 5 the general background and theorems are stated. Their proofs, however, require some extensive Fourier analysis. This is done in § 6. In § 7 we return to the proofs of the stated theorems.

Combining the results of Chapters I and II, we study the "Fourier-Laplace" transform for the group in Chapter III. This leads to our extension of the Hausdorff-Young theorem which is contained in § 8. In § 9, we complete the Fourier analysis of a function on the group by a consideration of the discrete series.

Chapter IV contains some applications of the above. In § 10 we are mainly concerned with the theorem that convolution by a function in  $L_p$ ,  $1 \leq p < 2$ , is a bounded operator on  $L_2$ . Some implications of this result are also deduced. Finally, in § 11, we deal with characterizations of various representations of the group and with a related notion—the "extendability" of a representation to  $L_p$ .

We should like to observe that, except for some notation, the contents of Chapters I and II are independent of each other. Since Chapter I is of a more technical nature, the reader might well begin with Chapter II which deals with uniformly bounded representation of the group.

<sup>3</sup> In the case of numerical valued functions this extension was obtained by one of us in [20].

## CHAPTER I. OPERATOR VALUED FUNCTIONS.

**2.  $L_p$  spaces of operator valued functions.** In this part we prove two purely technical theorems. In these results we have ignored various possible generalizations and have restricted our attention to a rather simple situation which appears to be adequate for our purpose.

We begin by introducing enough terminology to state the theorems.

Throughout the paper  $\mathcal{H}$  will denote a complex separable Hilbert space. The ring of all bounded operators on  $\mathcal{H}$  will be denoted by  $\mathcal{B}$ . If  $A$  is any non-negative operator in  $\mathcal{B}$  and  $\xi_1, \xi_2, \dots$  is any orthonormal basis of  $\mathcal{H}$ , then

$$(2.1) \quad \text{tr}(A) = \sum_{n=1}^{\infty} (A\xi_n, \xi_n)$$

is non-negative and independent of the choice of basis. The bound of an  $A$  in  $\mathcal{B}$  will be denoted by  $\|A\|_{\infty}$ , and we shall put  $|A| = (A^*A)^{1/2}$ . The  $p$ -th norm of  $A$  is then given by

$$(2.2) \quad \|A\|_p = (\text{tr}(|A|^p))^{1/p},$$

where  $1 \leq p < \infty$ . The letter  $M$  will always stand for a regular measure space<sup>4</sup> over a locally compact space with a countable basis for open sets; the underlying topological space will also be denoted by  $M$ . We shall consider functions on  $M$  whose values are bounded operators on  $\mathcal{H}$ . If  $F$  is such a function, we say that  $F$  is measurable provided

$$(2.3) \quad t \rightarrow (F(t)\xi, \eta), \quad t \in M,$$

is a measurable numerical function on  $M$  for each pair of vectors  $\xi, \eta$  in  $\mathcal{H}$ . If  $F, G$  are measurable, our assumptions imply the measurability of  $F + G$ ,  $FG$ , and  $F^*$ , these being defined in the obvious way; thus for example  $F^*$  is given by  $F^*(t) = [F(t)]^*$ .

An operator on  $\mathcal{H}$  is said to be of finite rank if it is reduced by a finite dimensional subspace. The set of all such operators is a two sided  $*$  ideal in  $\mathcal{B}$  and will be denoted by  $\mathcal{E}$ .

(2.4) By a *simple function* we shall mean a function  $F$  on  $M$  to  $\mathcal{E}$  having only a finite number of distinct values, each non-zero value being assumed on a set of finite measure.

<sup>4</sup> For a general discussion of measure theory on locally compact spaces see Halmos [10, Chapter 10].

**THEOREM 1.** *If  $F$  is a measurable operator valued function on  $M$ , the norms  $\|F(t)\|_p$ ,  $1 \leq p \leq \infty$ , are measurable as functions of  $t$ , and the relations*

$$(2.5) \quad \|F\|_\infty = \operatorname{ess\,sup}_{t \in M} \|F(t)\|_\infty$$

$$(2.6) \quad \|F\|_p = (\int \|F(t)\|_p^p dm(t))^{1/p}, \quad 1 \leq p < \infty,$$

*define norms relative to which the collection  $L_p(M, \mathfrak{B})$  of all measurable  $F$  with  $\|F\|_p < \infty$  is a complex Banach space (one identifies two functions if they differ only on null sets). Moreover, the formula*

$$(2.7) \quad \int F = \int \operatorname{tr} F(t) dm(t)$$

*is meaningful for  $F$  in  $L_1(M, \mathfrak{B})$  and defines an integral satisfying*

$$(2.8) \quad |\int F| \leq \|F\|_1.$$

**THEOREM 2.** *If  $F$  is any measurable operator valued function on  $M$  and  $1 \leq p \leq \infty$ , there exist simple functions  $S_1, S_2, \dots$  vanishing outside compact sets such that  $FS_n$  is integrable,*

$$(2.9) \quad \lim_n \int FS_n = \|F\|_p$$

*and  $\|S_n\|_q \leq 1$ ,  $1/p + 1/q = 1$ . Furthermore, if  $p$  and  $q$  are indices such that  $1/r = 1/p + 1/q \leq 1$  and  $F, G$  are measurable, then*

$$(2.10) \quad \|FG\|_r \leq \|F\|_p \|G\|_q.$$

*Finally, the simple functions vanishing outside compact sets are dense in  $L_p(M, \mathfrak{B})$  for all  $p$  such that  $1 \leq p < \infty$ .*

Except for the minor complication that we are dealing with operator valued functions, the proofs of these results involve nothing new and proceed along standard lines. We have nevertheless included most of the details for the benefit of the reader. We mention that one might obtain similar although less explicit theorems as consequences of known results from direct integral theory and the theory of non-commutative integration; however, it seems inappropriate to complicate an essentially simpler measure-theoretic situation by such considerations. Furthermore, in our application we require these results in the rather explicit and concrete form given above.

We begin by recalling some of the facts about the trace and the  $p$ -th norms mentioned earlier. As a general reference to this part, we refer to a paper [4] of Dixmier.

Let  $\xi, \eta$  be fixed vectors in  $\mathfrak{H}$  and define  $E_{\xi, \eta}$  by  $E_{\xi, \eta}(\zeta) = (\zeta, \eta)\xi$ ,  $\zeta \in \mathfrak{H}$ .

An operator  $E$  is of finite rank if and only if it is a finite sum of operators of the form  $E_{\xi, \eta}$  and moreover

$$(2.11) \quad \text{tr}(E_{\xi, \eta}) = (\xi, \eta).$$

Let  $\mathcal{B}_p$ ,  $1 \leq p \leq \infty$  denote the collection of all bounded operators  $A$  on  $\mathcal{H}$  such that  $\|A\|_p < \infty$ . A positive operator  $A$  is in  $\mathcal{B}_p$ ,  $1 \leq p < \infty$ , if and only if its spectrum  $\lambda_1, \lambda_2, \dots$  is discrete and

$$\sum_{n=1}^{\infty} \lambda_n^p < \infty.$$

In this event,  $\|A\|_p = (\sum_{n=1}^{\infty} \lambda_n^p)^{1/p}$ , which implies,

$$(2.12) \quad \|A\|_p \text{ is a non-decreasing function of } p.$$

$$(2.13) \quad \text{If } A \in \mathcal{B}_p, B \in \mathcal{B}_q, \text{ and } 1/r = 1/p + 1/q \leq 1, \text{ then } \|AB\|_r \leq \|A\|_p \|B\|_q, \text{ where } 1 \leq p, q \leq \infty.$$

$$(2.14) \quad \text{If } A \in \mathcal{B} \text{ and } 1 \leq p \leq \infty, \text{ there exist operators } E_1, E_2, \dots \text{ of finite rank such that } \|E_n\|_q \leq 1 \text{ and}$$

$$\lim_n \text{tr}(AE_n) = \|A\|_p,$$

where  $1/p + 1/q = 1$ . In case  $A$  is of finite rank, there exists another such operator  $E$  with  $\|E\|_q \leq 1$  and  $\text{tr}(AE) = \|A\|_p$ .

$$(2.15) \quad \mathcal{B}_p \text{ is a Banach space under the norm given by (2.2), and the collection } \mathcal{E} \text{ of operators of finite rank is dense in } \mathcal{B}_p \text{ for } 1 \leq p < \infty.$$

$$(2.16) \quad \text{If } 1 \leq p < \infty \text{ and } B \in \mathcal{B}_q, \text{ where } 1/p + 1/q = 1, \text{ then}$$

$$A \mapsto \text{tr}(AB), \quad A \in \mathcal{B}_p$$

is a bounded linear functional,  $\phi_B$  on  $\mathcal{B}_p$ , and the map  $B \mapsto \phi_B$  identifies  $\mathcal{B}_q$  with the conjugate space of  $\mathcal{B}_p$ .

Given an orthonormal basis  $\xi_1, \xi_2, \dots$  of  $\mathcal{H}$  we form the set  $\mathcal{D}$  of all finite rational linear combinations of the operators  $E_{ij}$  and  $(-1)^i E_{ij}$ , where

$$(2.17) \quad E_{ij} = E_{\xi_i, \xi_j}.$$

$\mathcal{D}$  is denumerable, and one easily verifies that the product of two members of  $\mathcal{D}$  is again in  $\mathcal{D}$ .

LEMMA 1.  $\mathcal{D}$  is dense in  $\mathcal{B}_p$ ,  $1 \leq p < \infty$ .

Suppose  $E$  is an operator of finite rank and that  $P$  is a projection of finite rank such that  $EP = E$ . Then for  $1 \leq p < \infty$  and  $A, B$  in  $\mathcal{D}$  we have

$$\|E - AB\|_p \leq \|EP - AB\|_1 \leq \|E - A\|_2 \|B\|_2 + \|A\|_2 \|P - B\|_2.$$

The inequalities follow from (2.12) and (2.13). Now because  $\mathcal{D}$  is dense in  $B_2$ , we see that any operator of finite rank can be approximated in the  $p$ -th norm by elements of  $\mathcal{D}$ . An application of (2.15) finishes the proof.

As a corollary we obtain the fact that  $\mathcal{B}_p$ ,  $1 \leq p < \infty$ , is separable.

The collection of all measurable operator valued functions on  $M$  will be denoted by  $\mathcal{B}(M)$ .

LEMMA 2. If  $F \in \mathcal{B}(M)$  and  $E$  is an operator of finite rank,

$$t \rightarrow \text{tr}(F(t)E)$$

is a measurable function on  $M$ .

There exist vectors  $\xi_1, \eta_1, \dots, \xi_n, \eta_n$  in  $\mathcal{H}$  such that

$$E = \sum_{i=1}^n E_{\xi_i, \eta_i}.$$

Thus  $F(t)E = \sum_{i=1}^n E_{F(t)\xi_i, \eta_i}$  and by (2.11),

$$\text{tr}(F(t)E) = \sum_{i=1}^n (F(t)\xi_i, \eta_i),$$

which implies  $\text{tr}(F(t)E)$  is measurable as a function of  $t$ .

LEMMA 3. If  $1 \leq p \leq \infty$  and  $F \in \mathcal{B}(M)$ ,  $t \rightarrow \|F(t)\|_p$  is a measurable function on  $M$ .

Let  $A$  belong to  $\mathcal{B}$ . From (2.13), (2.14), and Lemma 1 we see that

$$(2.18) \quad \|A\|_p = \text{Sup}\{|\text{tr}(AE)| : E \in \mathcal{D}, \|E\|_q \leq 1\},$$

where  $1/p + 1/q = 1$ . Replacing  $A$  by  $F(t)$  and applying Lemma 2 we see that  $\|F(t)\|_p$  is measurable; this follows from the fact that the least upper bound of a countable collection of measurable functions is again measurable.

LEMMA 4. A function  $F$  on  $M$  to  $\mathcal{B}_p$ ,  $1 \leq p < \infty$ , is measurable if and only if  $t \rightarrow \text{tr}(F(t)B)$  is measurable on  $M$  for all  $B$  in  $\mathcal{B}_q$  ( $1/p + 1/q = 1$ ).

For every pair of vectors  $\xi, \eta$  in  $\mathcal{H}$ ,  $B = E_{\xi, \eta}$  is in  $\mathcal{B}_q$ ; hence  $F \in \mathcal{B}(M)$ , provided  $\text{tr}(F(t)B)$  is measurable as a function of  $t$  for all  $B$  in  $\mathcal{B}_q$ .

Conversely, suppose  $F \in \mathcal{B}(M)$ . Let  $B \in \mathcal{B}_q$ . By (2.13)  $\text{tr}(F(t)B)$  exists and is finite for each  $t$  in  $M$ . If  $p = 1$ ,

$$\operatorname{tr}(F(t)B) = \lim_{n \rightarrow \infty} \sum_{i=1}^n (F(t)B_{\xi_i, \xi_i}),$$

where  $\xi_1, \xi_2, \dots$ , is any orthonormal basis of  $\mathcal{H}$ , and is therefore measurable. If  $1 < p < \infty$ , there exist, by (2.15), operators  $E_1, E_2, \dots$ , of finite rank such that  $\|B - E_n\|_q \rightarrow 0$ . Thus

$$|\operatorname{tr}(F(t)B) - \operatorname{tr}(F(t)E_n)| \leq \|F(t)\|_p \|B - E_n\|_q \rightarrow 0.$$

By Lemma 2,  $\operatorname{tr}(F(t)E_n)$  is measurable in  $t$  for each  $n$ , and hence  $\operatorname{tr}(F(t)B)$  is also.

The result just established together with (2.16) shows that a measurable function  $F$  on  $M$  to  $\mathcal{B}_p$  is weakly measurable as a function on  $M$  to the separable Banach space  $\mathcal{B}_p$ ; thus  $F$  is also strongly measurable.<sup>5</sup>

The proof of Theorem 1 now follows from the preceding lemmas and the well known theory<sup>5</sup> of the Lebesgue integral extended to functions with values in a Banach space.

In proving Theorem 2 it is convenient to establish

LEMMA 5. Suppose  $F = \sum_{i=1}^n f_i A_i$ , where  $f_i$  is a measurable numerical function on  $M$ ,  $f_i f_j = 0$ ,  $i \neq j$ , and  $A_i \in \mathcal{B}$ . Then for  $1 \leq p < \infty$ ,

$$(2.19) \quad \|F\|_p = \left( \sum_{i=1}^n \|f_i\|_{p^p} \|A_i\|_{p^p} \right)^{1/p},$$

and in case  $\|F\|_1 < \infty$ ,

$$(2.20) \quad \int F = \sum_{i=1}^n \left( \int f_i(t) dm(t) \right) (\operatorname{tr} A_i).$$

Since  $f_i(t)f_j(t) = 0$  ( $i \neq j$ ),

$$\begin{aligned} \|F\|_{p^p} &= \int \|F(t)\|_{p^p} dm(t) \\ &= \int \left( \sum_{i=1}^n |f_i(t)|^p \|A_i\|_{p^p} \right) dm(t) \\ &= \sum_{i=1}^n \left( \int |f_i(t)|^p dm(t) \right) \|A_i\|_{p^p}. \end{aligned}$$

If  $\|F\|_1 < \infty$ , then

$$\begin{aligned} \int F &= \int \operatorname{tr}(F(t)) dm(t) \\ &= \int \operatorname{tr} \left( \sum_{i=1}^n f_i(t) A_i \right) dm(t) \end{aligned}$$

<sup>5</sup> For a discussion of these points see Hille and Phillips [15, Chapter 3].

$$\begin{aligned} &= \int \left( \sum_{i=1}^n f_i(t) \operatorname{tr} A_i \right) dm(t) \\ &= \sum_{i=1}^n \left( \int f_i(t) dm(t) \right) \operatorname{tr} (A_i). \end{aligned}$$

LEMMA 6. *The simple functions vanishing outside compact sets are dense in  $L_p(M, \mathfrak{B})$  for  $1 \leq p < \infty$ .*

Let  $f, g$  be measurable numerical valued functions on  $M$ , and let  $A, B \in \mathfrak{B}$ . By simple estimates and the preceding lemma we have

$$(2.21) \quad \|fA - gB\|_p \leq \|f - g\|_p \|A\|_p + \|g\|_p \|A - B\|_p,$$

where  $1 \leq p < \infty$ . Thus if  $\epsilon > 0$ ,  $A \in \mathfrak{B}_p$ , and if  $f$  is the characteristic function of a set of finite measure, we can choose a compact set with characteristic function  $g$  and an operator  $B$  of finite rank such that

$$(2.22) \quad \|fA - gB\|_p < \epsilon.$$

The conclusion of the lemma follows from the fact<sup>5</sup> that finite linear combinations of functions of the form  $fA$  are dense in  $L_p(M, \mathfrak{B})$ .

LEMMA 7. *If  $F$  is a simple function with compact support there exists a simple function  $S$  with compact support such that  $\|S\|_q \leq 1$ , and*

$$(2.23) \quad \int FS = \|F\|_p,$$

where  $1 \leq p \leq \infty$ ,  $1/p + 1/q = 1$ .

There exist operators  $A_1, \dots, A_n$  of finite rank and mutually disjoint measurable subsets with characteristic functions  $f_1, \dots, f_n$  such that

$$F = \sum_{i=1}^n f_i A_i.$$

By (2.14) there exists an operator  $E_i$  of finite rank such that  $\|E_i\|_q \leq 1$  and  $\operatorname{tr}(A_i E_i) = \|A_i\|_p$ . Let

$$(2.24) \quad c_i = \|F\|_p^{1-p} \|A_i\|_p^{p-1}$$

and put

$$(2.25) \quad S = \sum_{i=1}^n c_i f_i E_i.$$

Then

$$\begin{aligned} \int FS &= \int \sum_{i=1}^n c_i f_i (A_i E_i) \\ &= \sum_{i=1}^n c_i \left( \int f_i(t) dm(t) \operatorname{tr}(A_i E_i) \right) = \sum_{i=1}^n c_i \|f_i\|_p^p \|A_i\|_p = \|F\|_p. \end{aligned}$$

Also

$$\begin{aligned}\|S\|_q^q &= \sum_{i=1}^n c_i^q \|f_i\|_q^q \|E_i\|_q^q \\ &\leq \sum_{i=1}^n \|F\|_p^{(1-p)q} \|A_i\|_p^{(p-1)q} \|f_i\|_p^p \\ &= \|F\|_p^{-p} \sum_{i=1}^n \|f_i\|_p^p \|A_i\|_p^p = 1.\end{aligned}$$

Finally, since  $F$  has compact support, so does  $S$ .

*Proof of Theorem 2.* Suppose  $F, G$  are measurable. To establish (2.10) we use (2.13) which implies

$$\begin{aligned}\|FG\|_r &\leq \int (\|F(t)\|_p \|G(t)\|_q)^r dm(t) \\ &\leq \int (\|F(t)\|_p^p dm(t))^{1/p} (\int \|G(t)\|_q^q dm(t))^{1/q}\end{aligned}$$

provided  $p \neq \infty$  and  $q \neq \infty$ . The other two cases arise treated by similar arguments.

As the case  $p = \infty$  is somewhat exceptional and requires separate treatment, we shall prove (2.9) only for  $p$  such that  $1 \leq p < \infty$ .<sup>\*</sup> Suppose first of all that  $\|F\|_p < \infty$ . By Lemma 6, there exist simple functions  $F_1, F_2, \dots$ , with compact supports such that  $\|F - F_n\|_p \rightarrow 0$ . Choose  $S_n$  for  $F_n$  in accordance with Lemma 7. By (2.10)  $FS_n$  is integrable and

$$|\int FS_n - \int F_n S_n| \leq \|F - F_n\|_p \rightarrow 0.$$

If  $\|F\|_p = \infty$ , let  $F_n$  be the product of  $F$  and the characteristic function of a set of finite measure contained in  $\{t: \|F(t)\|_p \leq n\}$  and chosen so that  $\|F_n\|_p \rightarrow \infty$ . Then  $\|F_n\|_p < \infty$ . Thus we can choose a simple function  $S_n$  with compact support contained in the support of  $F_n$  such that,  $\|S_n\|_q \leq 1$  and

$$|\int F_n S_n - \|F\|_p| < 1/n.$$

Then  $FS_n = F_n S_n$  and

$$\int FS_n \rightarrow \|F\|_p.$$

**3. Interpolation in the general case.** In this section we prove a rather general interpolation theorem for operator valued functions. Let  $\Phi$  be a complex valued function whose domain contains a strip,  $\alpha \leq Rz \leq \beta$ . We shall say that  $\Phi$  is *admissible* on the strip if  $\Phi$  is analytic in  $\alpha < Rz < \beta$ , continuous in  $\alpha \leq Rz \leq \beta$ , and satisfies the growth condition

<sup>\*</sup> We do not need the exceptional case  $p = \infty$  in our application.

$$(3.1) \quad \sup_{\alpha \leq x \leq \beta} \log |\Phi(x + iy)| \leq Ce^{\mu|y|},$$

where  $C$  and  $\mu$  are constants depending on  $\Phi$ ; we require also that  $\mu$  satisfies the additional condition

$$(3.2) \quad \mu < \pi/(\beta - \alpha).$$

If  $\Phi_1, \Phi_2$  are admissible on a given strip and if  $v_1, v_2$  are complex numbers it is easily verified that the combinations  $v_1\Phi_1 + v_2\Phi_2, \Phi_1\Phi_2$  are also admissible.

A complex valued function on a measure space will be called a *simple* function if it can be expressed as a finite linear combination of characteristic functions of measurable sets of finite measure.

Now let  $M_1, M_2$  be measure spaces, and let  $D$  be a strip,  $\alpha \leq Rz \leq \beta$ . Suppose  $B_z, z \in D$ , is a complex valued bilinear form defined for all simple functions  $f_1, f_2$  on  $M_1, M_2$ . We shall say that the collection  $\{B_z\}$  is an *admissible* family of bilinear forms on  $D$  if

$$(3.3) \quad \Phi(z) = B_z(f_1, f_2)$$

is admissible on  $D$  for each pair of simple functions,  $f_1, f_2$  on  $M_1, M_2$ .

We now introduce some notation and terminology which will remain fixed throughout this part.

The strip  $\alpha \leq Rz \leq \beta$  will be denoted by  $D$  and we shall put

$$(3.4) \quad \gamma = (1 - \tau)\alpha + \tau\beta, \quad 0 < \tau < 1.$$

We suppose  $p_0, p_1, q_0, q_1$  are given indices such that  $1 \leq p_i, q_i \leq \infty$ , and  $q_0 \neq \infty$  or  $q_1 \neq \infty$ . The indices  $p, q$  are then determined by

$$(3.5) \quad 1/p = (1 - \tau)1/p_0 + \tau 1/p_1,$$

$$(3.6) \quad 1/q = (1 - \tau)1/q_0 + \tau 1/q_1.$$

The conjugate indices of  $q_0, q_1, q$  will be denoted by  $q'_0, q'_1, q'$ . Finally,  $A_0, A_1$  will denote non-negative functions such that

$$(3.8) \quad \log A_i(y) \leq Ae^{\delta|y|}, \quad \delta < \pi/(\beta - \alpha).$$

With minor changes, the proof of Theorem 1 [20] yields the following convexity principle.

**LEMMA 8.** *Let  $\{B_z\}$  be an admissible family of bilinear forms on  $D$ , and suppose*

$$(3.9) \quad |B_{\alpha + i\nu}(f_1, f_2)| \leq A_0(y) \|f_1\|_{p_0} \|f_2\|_{q_0}$$

$$(3.10) \quad |B_{\beta, \alpha y}(f_1, f_2)| \leq A_1(y) \|f_1\|_{p_0} \|f_2\|_{q_1},$$

for all simple functions  $f_1, f_2$  on  $M_1, M_2$ . Then for simple functions  $f_1, f_2$  we also have

$$(3.11) \quad |\beta_\gamma(f_1, f_2)| \leq A_\gamma \|f_1\|_p \|f_2\|_{q'}.$$

The constant  $A_\gamma$  is given explicitly, in terms of the Poisson kernel for the strip, by

$$(3.12) \quad \log A_\gamma = \int_{-\infty}^{\infty} \log A_0[(\beta - \alpha)y] w(1 - \tau, y) dy + \int_{-\infty}^{\infty} \log A_1[(\beta - \alpha)y] w(\tau, y) dy,$$

where

$$w(\tau, y) = \frac{1}{2} \sin \pi / (\cosh \pi y + \cos \pi \tau).$$

By a bounded subset of a regular measure space we mean any measurable subset of a compact set. Now let  $N$  be an arbitrary measure space. Suppose  $T_z, z \in D$ , is a linear transformation from simple functions  $f$  on  $N$  to measurable operator valued functions  $T_z(f) = F_z$  on  $M$ . We shall say that  $\{T_z\}$  is an admissible family on  $D$  if  $(F_z(t)\xi, \eta)$  is locally integrable on  $M$  and

$$(3.13) \quad \Phi(z) = \int_K (F_z(t)\xi, \eta) dm(t)$$

is admissible on  $D$  for each choice of vectors  $\xi, \eta$  in  $\mathcal{H}$ , simple function  $f$  on  $N$ , and bounded subset  $K$  of  $M$ .

**THEOREM 3.** Let  $N$  be a measure space, and suppose  $\{T_z\}, z \in D$ , is an admissible family of linear transformation from simple functions  $f$  on  $N$  to measurable operator valued functions  $T_z(f) = F_z$  on  $M$ . Suppose further that the following two conditions are satisfied for each simple function  $f$ .

$$(3.14) \quad \|T_{\alpha, \alpha y}(f)\|_{q_0} \leq A_0(y) \|f\|_{p_0}.$$

$$(3.15) \quad \|T_{\beta, \alpha y}(f)\|_{q_1} \leq A_1(y) \|f\|_{p_1}.$$

Then it is also true that

$$(3.16) \quad \|T_\gamma(f)\|_q \leq A_\gamma \|f\|_p.$$

In proving the theorem it is convenient to establish the following lemma.

**LEMMA 9.** If  $\{T_z\}, z \in D$ , is an admissible family and  $S$  is a simple operator valued function on  $M$  which vanishes outside a compact set in  $M$ . Then  $\text{tr } F_z(t)S(t)$  is integrable and

$$(3.17) \quad \Phi(z) = \int \operatorname{tr} F_z(t) S(t) \, dm(t)$$

$f_1, f_2$  is admissible on  $D$  for each simple  $f$  on  $N$ .

Suppose first that  $S = kE_{\xi, \eta}$ , where  $k$  is the characteristic function of a bounded set. Then

$$\operatorname{tr} F_z(t) S(t) = k(t) (F_z(t) \xi, \eta),$$

and the result follows by assumption. The general case follows by linearity.

*Proof of the theorem.* Our assumptions imply  $q \neq \infty$ .<sup>7</sup> Thus to show that

$$\|T_\gamma(f)\|_q \leq A_\gamma \|f\|_p$$

it suffices, in view of Theorem 2, to show that

$$(3.18) \quad \left| \int \operatorname{tr} F_\gamma(t) S(t) \, dm(t) \right| \leq A_\gamma \|f\|_p \|S\|_q$$

for each simple function  $S$  vanishing outside a compact set.

The idea of the proof is to reduce this problem to one concerning an admissible family of bilinear forms. We shall then apply Lemma 8 to complete the argument.

Suppose then that  $S$  is a simple function with compact support. We can express  $S$  as  $\sum_{i=1}^n k_i E_i$ , where  $k_1, k_2, \dots, k_n$  are the characteristic functions of mutually disjoint bounded subsets  $K_i$  and each  $E_i$  is an operator of finite rank. Now let  $E_i = U_i |E_i|$  be the canonical polar decomposition of  $E_i$ , and let

$$(3.19) \quad \sum_j \lambda_{ij} P_{ij}, \quad \lambda_{ij} > 0,$$

be the spectral decomposition of  $|E_i|$ . The pairs of indices  $i, j$  will then range over a finite set which we shall call  $M_2$ . To each complex valued function  $g = \{g_{ij}\}$  defined on  $M_2$  we associate an operator valued function  $G$  on  $M$  which is given by

$$(3.20) \quad G(t) = \sum_{i=1}^n k_i(t) \sum_j g_{ij} U_i P_{ij}.$$

Then  $G$  is a simple function with compact support, and by an elementary computation we get

$$(3.21) \quad G^*(t) G(t) = \sum_{i,j} k_i(t) |g_{ij}|^2 P_{ij}.$$

<sup>7</sup> The case  $q = \infty$  could be dealt with by a more involved argument.

Now for  $1 \leq p < \infty$ ,  $\|G\|_p^p = \int \text{tr}[(G^*(t)G(t))^{p/2}] dm(t)$  which implies

$$(3.22) \quad \|G\|_p = \left( \sum_{i,j} |g_{ij}|^p \|k_i\|_p^p \right)^{1/p}.$$

Since  $\|k_i\|_p^p$  is independent of  $p$  being, in fact, equal to the measure of  $K_i$ , we can introduce a measure in  $M_2$  relative to which  $\|g\|_p = \|G\|_p$ . We observe that this relation is also valid for  $p = \infty$ . Because the maps  $f \rightarrow F_z$ ,  $g \rightarrow G$  are linear it follows that the equation

$$(3.23) \quad B_z(f, g) = \int \text{tr}(F_z(t)G(t)) dm(t)$$

defines a bilinear form for each  $z$  in  $D$ . In this formula  $f$  is an arbitrary simple function on  $M_1 = N$ , and  $g$  is any complex valued, obviously simple, function on  $M_2$ . By Lemma 9, in particular by (3.17),  $\{B_z\}$  is an admissible family on  $D$ . Now

$$|B_{\alpha+iy}(f, g)| \leq \|F_{\alpha+iy}G\|_1 \leq \|F_{\alpha+iy}\|_{q_0} \|G\|_{q_0'}$$

and using (3.14) we get

$$(3.24) \quad |B_{\alpha+iy}(f, g)| \leq A_0(y) \|f\|_{p_0} \|g\|_{q_0'}.$$

By similar estimates we obtain

$$(3.25) \quad |B_{\beta+iy}(f, g)| \leq A_1(y) \|f\|_{p_1} \|g\|_{q_1'}.$$

Thus by Lemma 8,  $|B_\gamma(f, g)| \leq A_\tau \|f\|_p \|g\|_{q'}$ . As this holds for all  $g$ , we may take  $g = \{\lambda_{ij}\}$ . Then  $G = S$  and

$$(3.26) \quad B_\gamma(f, g) = \int \text{tr} F_\gamma(t) S(t) dm(t),$$

which implies (3.18).

**4. The main interpolation theorem.** In order to prove our results for the  $2 \times 2$  real unimodular group, we use, in addition to facts about the group and its representations, certain convexity arguments. The basic and most important fact along these lines is established in this section; with the intent of clarifying the situation, we have presented it in a slightly more general form than our application requires.

An operator valued function  $\mathcal{F}$  defined on an open strip,  $\alpha_0 < Rs < \beta_0$ , in the complex  $s$ -plane is said to be *analytic* if  $(\mathcal{F}(s)\xi, \eta)$  is analytic for all  $\xi, \eta$  in  $\mathcal{H}$ . We shall say that  $\mathcal{F}$  is of *admissible growth* in the strip if

$$(4.1) \quad \sup_{\alpha_0 < \sigma < \beta_0} \log \|\mathcal{F}(\sigma + it)\|_\infty \leq C e^{\mu|t|}, \quad \mu < \pi/(\beta_0 - \alpha_0).$$

**THEOREM 4.** Let  $N$  be a measure space and  $T$  be a linear map from

simple functions on  $N$  to analytic operator valued functions such that  $\mathcal{F} = Tf$  is of admissible growth on the strip  $\alpha_0 < \text{Re } s < \beta_0$  for each simple function  $f$ . Suppose that for  $\alpha_0 < \alpha < \beta < \beta_0$  we have

$$(4.2) \quad \sup_{-\infty < t < \infty} \|\mathcal{F}(\alpha + it)\|_{\infty} (1 + |t|)^c \leq A_0 \|f\|_1,$$

$$(4.3) \quad \left( \int_{-\infty}^{\infty} \|\mathcal{F}(\beta + it)\|_2^2 |t|^{2a} (1 + |t|)^{2b} dt \right)^{\frac{1}{2}} \leq A_1 \|f\|_2$$

for all simple  $f$ , where  $a, b, c$  are real and  $a \geq 0$ . Then we may conclude

$$(4.4) \quad \left( \int_{-\infty}^{\infty} \|\mathcal{F}(\gamma + it)\|_q^q (1 + |t|)^{qd} dt \right)^{1/q} \leq A_{\tau} \|f\|_p,$$

where  $1 < p < 2$ ,  $1/p + 1/q = 1$ ,  $\gamma = \alpha + \tau(\beta - \alpha)$ ,  $d = c + \tau(a + b - c)$ , and the parameter  $\tau$  is determined by  $1/p = 1 - \tau/2$ .

*Remarks.* Before we prove the theorem, we notice that the result (4.4) is intermediate—in the sense of Riesz-Thorin convexity—between the hypotheses (4.2) and (4.3). It should be noted that the singularity at  $t = 0$  of the measure  $|t|^{2a} (1 + |t|)^{2b} dt$  does not persist in the conclusion; only the influence of  $|t|^{2a}$  for  $t$  near infinity remains.

The proof given below could be generalized in several directions. We may begin with a general pair of indices  $(p_0, q_0)$ ,  $(p_1, q_1)$  instead of  $(1, \infty)$  and  $(2, 2)$ . We might also consider more general measures than those of the form  $|t|^{2a} (1 + |t|)^{2b} dt$  given above. We shall not consider these generalizations here.

It should be pointed out that the proof given below would be much simpler if  $a, b$ , and  $c$  were zero. In that case the left-hand sides of (4.2) and (4.3) would be translation invariant in  $t$ . Since the basic method of the proof consists of translation along vertical lines of the strip, we are forced to overcome the lack of translation invariance by somewhat complicated devices.

At several points in the proof it will be convenient to refer to the easily verified result given below:

LEMMA 10. If  $\nu$  is real and  $\delta > 0$ , there exists a constant  $A > 0$  such that

$$(4.5) \quad (\delta + |y + t|)^{\nu} \leq A(1 + |y|)^{|\nu|} (1 + |t|)^{\nu}$$

for  $-\infty < y, t < \infty$ .

*Proof of the theorem.* We shall obtain the result as a consequence of Theorem 3. To do this we set  $M = (-\infty, \infty)$  and put

$$(4.5) \quad dm = (1 + |t|)^{2(a+b-c)} dt,$$

where  $dt$  is Lebesgue measure. Given a simple function  $f$  on  $N$  we form  $\mathcal{F} = Tf$  and set

$$(4.6) \quad F_z(t) = \mathcal{F}(z + it)(1 + |t|)^{c-a}(z - \beta + it)^a$$

for  $\alpha \leq Rz \leq \beta$ . Since  $a \geq 0$ , we may choose a single valued branch of the factor  $(z - \beta + it)^a$  which is analytic in  $\alpha < Rz < \beta$  and continuous on  $\alpha \leq Rz \leq \beta$ . Thus  $(F_z(t), \eta)$  is analytic in  $z$  for each  $t$  and is jointly continuous in  $z, t$  for all vectors  $\xi, \eta$  in  $\mathcal{H}$ . Furthermore, the transformation  $T_z$  defined by  $T_z(f) = F_z$  is linear and maps simple functions on  $N$  to measurable operator valued functions on  $M$ .

We shall now estimate  $\|F_z(t)\|_\infty$ . By (4.6) and the condition (4.1) that  $\mathcal{F}$  is of admissible growth in  $\alpha \leq Rz \leq \beta$ , we find,

$$(4.7) \quad \begin{aligned} \|F_z(t)\|_\infty &= \|\mathcal{F}(x + i(y+t))\|_\infty (1 + |t|)^{c-a} |x - \beta + i(y+t)|^a \\ &\leq A \|\mathcal{F}(x + i(y+t))\|_\infty (1 + |t|)^c (1 + |y|)^a \end{aligned}$$

Hence,

$$\log \|F_z(t)\|_\infty \leq Ce^{\mu|y+t|} + \log(1 + |t|)^c + \log(1 + |y|)^a + \log A.$$

This estimate together with the above implies the condition (3.13) that  $\{T_z\}$  be an admissible family. Now, for  $z = \alpha + iy$  we find, using (4.2), (4.7), that

$$\begin{aligned} \|F_{\alpha+iy}(t)\|_\infty &\leq A \|\mathcal{F}(\alpha + i(y+t))\|_\infty (1 + |t|)^c (1 + |y|)^a \\ &\leq A \|f\|_1 (1 + |y+t|)^{-c} (1 + |t|)^c (1 + |y|)^a. \end{aligned}$$

Thus by Lemma 10, we obtain (4.8).

$$(4.8) \quad \|T_{\alpha+iy}(f)\|_\infty \leq A(1 + |y|)^{|a-c|} \|f\|_1.$$

Next we shall estimate  $\|T_{\beta+iy}(f)\|_2$ . We have,

$$\|F_{\beta+iy}\|_2^2 = \int_{-\infty}^{\infty} \|\mathcal{F}(\beta + i(y+t))\|_2^2 |y+t|^{2a} (1 + |t|)^{2c-2a} dm.$$

Now making use of (4.3), we obtain

$$\begin{aligned} \|F_{\beta+iy}\|_2^2 &\leq (A_1 \|f\|_2)^2 \sup_{-\infty < t < \infty} [(1 + |y+t|)^{-2b} (1 + |t|)^{2c-2a} (1 + |t|)^{2(a+b-c)}] \\ &\leq (A_1 \|f\|_2)^2 \sup_{-\infty < t < \infty} [(1 + |y+t|)^{-2b} (1 + |t|)^{2b}]. \end{aligned}$$

Thus by Lemma 10,

$$(4.9) \quad \|F_{\beta+iy}\|_2 \leq A(1 + |y|)^{|b|} \|f\|_2.$$

Having (4.8) and (4.9) we can apply Theorem 3 and conclude that

$$(4.10) \quad \|T_\gamma(f)\|_q \leq A_\tau \|f\|_p.$$

Now

$$\begin{aligned} \|T_\gamma(f)\|_{q^q} &= \int_{-\infty}^{\infty} \|\mathcal{F}(\gamma + it)\|_{q^q} (1 + |t|)^{qc-qa} |\gamma - \beta + it|^{qa} dm \\ &\leq A_\tau \|f\|_{p^q}. \end{aligned}$$

Hence

$$\int_{-\infty}^{\infty} \|\mathcal{F}(\gamma + it)\|_{q^q} (1 + |t|)^{qc-qa} (1 + |t|)^{qa} (1 + |t|)^{2(a+b-c)} dt \leq A \|f\|_{p^q}.$$

Since  $\tau q = 2$ ,

$$qd = gc + q\tau(a + b - c) = qc + 2(a + b - c).$$

Thus

$$\int_{-\infty}^{\infty} \|\mathcal{F}(\gamma + it)\|_{q^q} (1 + |t|)^{qd} dt \leq A \|f\|_{p^q},$$

which proves the theorem.

**5. Uniformly bounded representations.** We now consider the group  $G$  of  $2 \times 2$  real unimodular matrices, and we first recall some of the known facts concerning the representations of  $G$ .

We represent an element  $g \in G$ , by

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad ad - bc = 1,$$

and denote by  $g(x)$  the fractional linear transformation

$$g(x) = (ax + c)/(bx + d), \quad -\infty < x < \infty.$$

Then

$$(5.1) \quad (g_1 g_2)(x) = g_2(g_1 x)$$

and  $dg(x)/dx = (bx + d)^{-2}$ ,  $bx + d \neq 0$ .

We now introduce two "multipliers"  $\phi^+$  and  $\phi^-$ . These are defined by

$$(5.2) \quad \phi^+(g, x, s) = |bx + d|^{2s-2}$$

$$(5.3) \quad \phi^-(g, x, s) = \operatorname{sgn}(bx + d) \phi^+(g, x, s),$$

where  $s$  is an arbitrary complex number.

Next we consider the "multiplier representations"

$$g \rightarrow v^{\pm}(g, s)$$

given for functions  $f$  on the real axis by

$$(5.4) \quad v^{\pm}(g, s) : f(x) \rightarrow \phi^{\pm}(g, x, s)f(g(x)).$$

From these, one may obtain the irreducible unitary representations of  $G$ . They fall into three classes.<sup>\*</sup>

a) The two *continuous principal series*

$$g \rightarrow v^{\pm}(g, 1/2 + it), \quad -\infty < t < \infty,$$

where the Hilbert space is the space  $L_2$  of square integrable functions on  $-\infty < x < \infty$ , with the usual measure.

b) The *complementary series*

$$g \rightarrow v^{\pm}(g, \sigma), \quad 0 < \sigma < 1/2.$$

The Hilbert space, in this case, is defined by the inner product

$$(5.5) \quad (f, h)_{\sigma} = a_{\sigma} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) h(y) |x - y|^{-2\sigma} dx dy,$$

where  $a_{\sigma} = \Gamma(2\sigma) \cos(\sigma\pi) / \pi$ .

c) The two *discrete series*,

$$g \rightarrow D^{\pm}(g, k), \quad k = 0, 1, 2, \dots$$

We shall not need the exact form of these representations.

The Plancherel formula for  $G$  was derived by Harish-Chandra [13]. It involves representations of type a) and c) and not of type b). To state it we make the usual definition

$$U(f) = \int_G f(g) U(g) dg$$

for uniformly bounded representations  $g \rightarrow U(g)$  and  $f$  in  $L_1(G)$ . Using this notation the Plancherel formula asserts that, whenever  $f \in L_1(G) \cap L_2(G)$ ,

$$\begin{aligned} \|f\|_2^2 = & 1/2 \int_{-\infty}^{\infty} \|v^+(f, 1/2 + it)\|_2^2 t \tanh \pi t dt \\ & + 1/2 \int_{-\infty}^{\infty} \|v^-(f, 1/2 + it)\|_2^2 t \coth \pi t dt \end{aligned}$$

\* Except for notation, these representations are those of Bargmann [1]; the difference of notation is discussed more fully in the proof of Theorem 10 in § 11.

$$\begin{aligned}
 & + \sum_{k=0}^{\infty} \|D^+(f, k)\|_2^2 (k + 1/2) \\
 & + \sum_{k=0}^{\infty} \|D^-(f, k)\|_2^2 (k + 1).
 \end{aligned}$$

Here  $\|\cdot\|_2$  means the usual Hilbert-Schmidt norm for operators as used in § 2 above.

One of our main results is contained in the following theorem.

**THEOREM 5.** *There exists a separable Hilbert space  $\mathfrak{H}$  and representations*

$$g \rightarrow U^z(g, s)$$

of  $G$  on  $\mathfrak{H}$  with the following properties:

1)  $g \rightarrow U^z(g, s)$  is a continuous representation of  $G$  on  $\mathfrak{H}$  for each complex  $s$  in the strip  $0 < R(s) < 1$ .

2)  $g \rightarrow U^z(g, 1/2 + it)$  is unitarily equivalent to the representation  $g \rightarrow v^z(g, 1/2 + it)$  of the continuous principal series defined above.

3)  $g \rightarrow U^+(g, \sigma)$ ,  $0 < \sigma < 1/2$  is unitarily equivalent to the representation  $g \rightarrow v^+(g, \sigma)$  of the complementary series.

4) If  $\xi$  and  $\eta$  are two vectors in  $\mathfrak{H}$ , then the functions

$$s \rightarrow (U^z(g, s)\xi, \eta), \quad g \text{ fixed,}$$

are analytic in  $0 < R s < 1$ .

$$5) \quad \sup_g \|U^z(g, s)\|_x \leq A_\sigma (1 + |t|)^{\frac{1}{2}},$$

$s = \sigma + it$ ,  $0 < \sigma < 1$ .<sup>9</sup> Furthermore, the constant  $A_\sigma$  is bounded on any interval of the form  $0 < \alpha \leq \sigma \leq \beta < 1$ .

It is known that for each  $t$ , the representations  $v^z(\cdot, 1/2 + it)$  and  $v^z(\cdot, 1/2 - it)$  are unitarily equivalent. Hence the same fact holds for the representations  $U^z(\cdot, 1/2 + it)$  and  $U^z(\cdot, 1/2 - it)$ .

As the next theorem shows, these equivalences are to some extent already inherent in the "analytic structure" of the representations  $g \rightarrow U^z(g, s)$ ; the theorem also describes some additional, and rather interesting, relations among the representations  $U^z(\cdot, s)$ .

**THEOREM 6.** *The following symmetries exist:*

<sup>9</sup> As in § 2, the ordinary bound of an operator  $A$  is denoted by  $\|A\|_x$ .

1) The representations  $U^+(\cdot, s)$  and  $U^+(\cdot, 1-\bar{s})$  are contragredient. Similarly,  $U^-(\cdot, s)$  and  $U^-(\cdot, 1-\bar{s})$  are contragredient.

2)  $U^+(\cdot, s) = U^+(\cdot, 1-s)$ . Hence  $U^+(\cdot, s)$  and  $U^+(\cdot, \bar{s})$  are also contragredient.

3) There exists a fixed non-scalar unitary operator  $S$  such that for all  $s$  in  $0 < Rs < 1$ ,

$$SU^-(\cdot, s)S^{-1} = U^-(\cdot, 1-s).$$

Thus  $U(\cdot, \bar{s})$  is unitarily equivalent to the contragredient of  $U^-(\cdot, s)$ .

*Remarks.* (i) It should be observed that the known result [1] concerning the reducibility of the representation  $U^-(\cdot, 1/2)$  is implied by 3).

(ii) The representations  $U^+(\cdot, s)$  for  $s \neq 1/2 + it$  and  $s \neq \sigma$  are unitarily equivalent to representations introduced by Mautner and Ehrenpreis [5]. These they show are not equivalent to unitary ones. They also assert that the representations are uniformly bounded. However, the more definite statement contained in 5) of Theorem 5 is crucial for our purposes.

(iii) The proof of Theorem 5 is lengthy and requires some vigorous classical Fourier analysis. This is contained in § 6, which is, for the most part, somewhat technical. At first reading the reader may prefer to pass on to § 7.

**6. Some lemmas from Fourier analysis.** We shall begin by introducing a class of Hilbert spaces, which will be seen<sup>10</sup> to be related to the  $L_p$  spaces via the Fourier transform. These spaces  $\mathcal{H}_\sigma$ , are indexed by a parameter  $0 < \sigma < 1$ , and are given by the norm

$$(6.1) \quad \|F\|_\sigma = \left( \int_{-\infty}^{\infty} |F(x)|^2 |x|^{2\sigma-1} dx \right)^{1/2}.$$

The spaces  $\mathcal{H}_{\sigma_1}, \mathcal{H}_{\sigma_2}$  corresponding to any pair of indices  $\sigma_1, \sigma_2$ , such that  $0 < \sigma_1, \sigma_2 < 1$ , are naturally related by a family of unitaries which we shall now exhibit. Let  $s_1 = \sigma_1 + it_1$  and  $s_2 = \sigma_2 + it_2$  where  $-\infty < t_1, t_2 < \infty$ . Now let  $W(s_1, s_2)$  be the mapping with the domain  $\mathcal{H}_{\sigma_1}$  given by

$$(6.2) \quad F(x) \rightarrow F(x) |x|^{s_1-s_2}, \quad F \in \mathcal{H}_{\sigma_1}.$$

<sup>10</sup> Although many of the results of this section are probably known, they do not seem to be accessible in the literature in the manner in which we need them.

Then

$$\|W(s_1, s_2)F\|_{\sigma_2}^2 = \int_{-\infty}^{\infty} |F(x)|^2 |x|^{2(\sigma_1 - \sigma_2)} |x|^{2\sigma_2 - 1} dx = \|F\|_{\sigma_1}^2.$$

This fact together with (6.2) shows that  $W(s_2, s_1)$  is the inverse of  $W(s_1, s_2)$ .

In what follows, we shall be mainly concerned with the pair of spaces  $\mathcal{H}_\sigma$  and  $\mathcal{H}_{1-\sigma}$ . For the sake of convenience we shall set  $W_s = W(s, 1-s)$ . The mapping  $W_\sigma$  is of particular interest because it implements a duality between  $\mathcal{H}_\sigma$  and  $\mathcal{H}_{1-\sigma}$ . In order to make this statement precise, we shall introduce some additional notation. Throughout this section and the one that follows, it will frequently be convenient to put

$$(6.3) \quad (F, G) = \int_{-\infty}^{\infty} F(x) \overline{G(x)} dx.$$

This notation will be used with the understanding that  $F, G$  are measurable complex valued functions defined on  $-\infty < x < \infty$  such that  $FG$  is integrable. The inner product in  $\mathcal{H}_\sigma$  will be denoted by  $(\cdot, \cdot)_\sigma$ , and we shall sometimes set  $1 - \sigma = \sigma'$ .

LEMMA 11. If  $F \in \mathcal{H}_\sigma$  and  $G \in \mathcal{H}_{\sigma'}$ ,  $0 < \sigma < 1$ , then

$$(6.4) \quad (W_\sigma F, G)_{\sigma'} = (F, G) = (F, W_{\sigma'} G)_\sigma,$$

$$(6.5) \quad |(F, G)| \leq \|F\|_\sigma \|G\|_{\sigma'}.$$

Furthermore, if  $A$  is a bounded operator on  $\mathcal{H}_\sigma$ , the operator

$$(6.6) \quad A' = W_\sigma A^* W_{\sigma'}^{-1},$$

where  $A^*$  is the (Hilbert space) adjoint of  $A$  is characterized as the unique bounded operator on  $\mathcal{H}_{\sigma'}$  such that

$$(6.7) \quad (A(F), G) = (F, A'(G))$$

for all  $F$  in  $\mathcal{H}_\sigma$  and all  $G$  in  $\mathcal{H}_{\sigma'}$ .

To prove (6.4) we first observe that  $2\sigma' - 1 = 1 - 2\sigma$ . Thus

$$\begin{aligned} (W_\sigma F, G)_{\sigma'} &= \int_{-\infty}^{\infty} F(x) |x|^{2\sigma-1} \overline{G(x)} |x|^{1-2\sigma} dx \\ &= (F, G) \\ &= \int_{-\infty}^{\infty} F(x) \overline{G(x)} |x|^{1-2\sigma} |x|^{2\sigma-1} dx \\ &= (F, W_{\sigma'} G)_\sigma. \end{aligned}$$

Now, by Schwartz's inequality,

$$|(F, G)| \leq \|W_\sigma F\|_{\sigma'} \|G\|_{\sigma'},$$

and (6.5) follows from the fact that  $W_\sigma$  is an isometry. Suppose  $A$  is a bounded operator on  $\mathcal{H}_\sigma$ , and that  $F \in \mathcal{H}_\sigma$ ,  $G \in \mathcal{H}_{\sigma'}$ . By (6.4), the fact that  $W_\sigma$  preserves inner products, and a second application of (6.4) we find that

$$\begin{aligned} (A(F), G) &= (A(F), W_\sigma G)_\sigma \\ &= (F, A^* W_\sigma G)_\sigma = (W_\sigma F, W_\sigma A^* W_\sigma G)_{\sigma'} \\ &= (F, W_\sigma A^* W_\sigma G). \end{aligned}$$

Thus (6.7) is satisfied by the operator  $A' = W_\sigma A^* W_\sigma^{-1}$ . That  $A'$  is the unique operator with this property follows from the easily established fact that,  $G \in \mathcal{H}_{\sigma'}$  and  $(F, G) = 0$  for all  $F \in \mathcal{H}_\sigma$  implies  $G(x) = 0$  a.e. It should also be observed that  $(A')' = A$ .

In addition to the  $\mathcal{H}_\sigma$  spaces we shall consider the  $L_p$  spaces,  $1 \leq p < \infty$ , of functions  $f$  defined on  $-\infty < x < \infty$  and normed by

$$\|f\|_p = \left( \int_{-\infty}^{\infty} |f(x)|^p dx \right)^{1/p}.$$

Since the parameter  $\sigma$  ranges between 0 and 1 and  $1 \leq p < \infty$  there should be no confusion between the norms  $\|\cdot\|_\sigma$ , and  $\|\cdot\|_p$ .

For a function  $f$  defined on  $-\infty < x < \infty$ , the Fourier transform  $F$  is defined by

$$(6.6) \quad F(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-ixy} f(y) dy$$

and the inverse Fourier transform is given by

$$(6.7) \quad f(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{ixy} F(y) dy.$$

Here and throughout this section we shall adhere to the following convention. Pairs of functions which are related to each other by either (6.6) or (6.7) will be denoted by corresponding lowercase and capital letters such as  $f, F$  or  $g, G$ . Furthermore, we take for granted such standard facts as the Plancherel theorem, and the sense in which these transforms exist for functions in  $L_p$ ,  $1 \leq p \leq 2$ , as well as the equivalence of (6.6) with (6.7) under suitable restrictions on  $f$  or  $F$ . (See e.g. [21]). To be more specific, we shall make use of two well known results on  $L_p$  transforms, a theorem of Titchmarsh (the so-called Hausdorff-Young theorem), and the Parseval formula for  $L_p, L_q$ . These results may be stated as follows:

LEMMA 12. If  $f, G \in L_p$ ,  $1 \leq p \leq 2$ , and their Fourier transforms  $F, g$  are given by (6.6), (6.7), then

$$a) \quad \|F\|_q \leq A \|f\|_p,$$

$$b) \quad \|g\|_q \leq A \|G\|_p,$$

where  $1/p + 1/q = 1$ , and

$$c) \quad (f, g) = (F, G).$$

Now the relation between the  $L_p$  spaces and the  $\mathcal{H}_\sigma$  spaces mentioned earlier is contained in the following lemma.

LEMMA 13. Let  $f \in L_p$ ,  $1 < p \leq 2$ , and let  $F$  denote its Fourier transform (6.6). Let  $\sigma = 1 - 1/p$ . Then  $0 < \sigma \leq 1/2$ ,  $F \in \mathcal{H}_\sigma$ , and

$$(6.8) \quad \|F\|_\sigma \leq A_\sigma \|f\|_p, \quad 1 < p \leq 2.$$

The class of  $F \in \mathcal{H}_\sigma$  which are Fourier transforms of  $f \in L_p$  is dense in  $\mathcal{H}_\sigma$ ;  $1 < p \leq 2$ ,  $0 < \sigma \leq 1/2$ .

Analogously, let  $F \in \mathcal{H}_\sigma$ ,  $1/2 \leq \sigma < 1$ , and let  $\sigma = 1 - 1/p$ . Then  $2 \leq p < \infty$ , the inverse transform (6.7) exists in  $L_p$  norm, and

$$\|f\|_p \leq A_\sigma \|F\|_\sigma, \quad 1/2 \leq \sigma < 1.$$

Consider first the case  $1 < p \leq 2$ . By a theorem of Hardy and Littlewood (see [11], p. 375),

$$\left( \int_{-\infty}^{\infty} |F(x)|^p |x|^{p-2} dx \right)^{1/p} \leq A_p \|f\|_p, \quad 1 < p \leq 2.$$

Now,

$$\begin{aligned} \|F\|_{\sigma^2} &= \int_{-\infty}^{\infty} |F(x)| \cdot |F(x)| |x|^{2\sigma-1} dx \\ &\leq \left( \int_{-\infty}^{\infty} |F(x)|^p dx \right)^{1/p} \left( \int_{-\infty}^{\infty} (|F(x)| |x|^{2\sigma-1})^p dx \right)^{1/p}, \end{aligned}$$

by Holder's inequality. Furthermore,  $(2\sigma - 1)p = p - 2$ . Thus using the inequalities of Titchmarsh (Lemma 12) and Hardy and Littlewood we obtain

$$\|F\|_{\sigma^2} \leq A_p \|f\|_p^2, \quad 1 < p \leq 2.$$

This proves (6.8).

If  $F$  is the characteristic function of a finite interval, then  $f$  given by (6.7) is in  $L_p$  for all  $p > 1$ . Hence finite linear combinations of characteristic functions of finite intervals are contained among the Fourier transforms (6.6) of  $f \in L_p$ . Therefore the image of  $L_p$ ,  $1 < p \leq 2$ , under the Fourier

transform is dense in the corresponding space  $\mathcal{H}_\sigma$ ,  $\sigma = 1 - 1/p$ . This concludes the consideration of the case  $1 < p \leq 2$ .

The second part of the lemma, which deals with the case  $1/2 \leq \sigma < 1$ ,  $2 \leq p < \infty$ , follows from the first part by duality. We shall briefly indicate the argument. Put  $\sigma' = 1 - \sigma$ . Then  $\sigma' = 1 - 1/p'$ , where  $1/p + 1/p' = 1$  and  $1 < p' \leq 2$ . By Lemma 11,  $\mathcal{H}_{\sigma'}$  and  $\mathcal{H}_\sigma$  are dual and it is well known that  $L_{p'}$  and  $L_p$  are dual. The second part of the lemma then follows upon identifying (6.7) with the adjoint of (6.6) considered as a mapping from  $L_{p'}$  to  $\mathcal{H}_{\sigma'}$  (properly speaking, this can be done only on a dense subset of  $\mathcal{H}_{\sigma'}$ ).

LEMMA 14. *Let*

$$K(F)(x) = \int_{-\infty}^{\infty} |1 - |x/y||^\alpha |x-y|^{-1} F(y) dy.$$

Then

$$\int_{-\infty}^{\infty} |K(F)(x)|^2 dx \leq A_\alpha \int_{-\infty}^{\infty} |F(x)|^2 dx$$

if  $-1/2 < \alpha < 1/2$ .

This lemma is known. The proof follows easily from Theorem 319 of [12]. There, a more general theorem on integral operators whose kernels are homogeneous of degree  $-1$  is given.

The main discussion of this section is contained in the lemma below. We shall deal with operators acting on  $\mathcal{H}_\sigma$ . It will be convenient, however, to specify the action of these operators by exhibiting their action on the Fourier transforms of the functions in question.

Thus we consider the multiplication operators

$$(6.9) \quad m_i^+ : f(x) \rightarrow |x|^{2it} f(x)$$

$$(6.10) \quad m_i^- : f(x) \rightarrow \operatorname{sgn}(x) |x|^{2it} f(x).$$

Now if  $F$  is the Fourier transform of  $f \in L_1 \cap L_2$ , we shall denote the Fourier transforms of  $m_i^+(f)$ ,  $m_i^-(f)$  by  $M_i^+(F)$ ,  $M_i^-(F)$ .

It will also be convenient to introduce the following class  $\mathcal{D}$  of functions:  $F \in \mathcal{D}$  if  $F$  is  $C^\infty$  and vanishes in a neighborhood of zero and outside a compact set. Clearly  $\mathcal{D}$  is dense in each  $\mathcal{H}_\sigma$ ,  $0 < \sigma < 1$ . Furthermore,  $\mathcal{D}$  is contained in the image of  $L_1 \cap L_2$  under the Fourier transform, (6.6).

LEMMA 15. *If  $F \in \mathcal{D}$ , then  $M_i^\pm(F) \in \mathcal{H}_\sigma$  for each  $\sigma$  such that  $0 < \sigma < 1$ , and the transformations*

$$F \rightarrow M_i^\pm(F), \quad F \in \mathcal{D}$$

*have unique bounded extensions to all of  $\mathcal{H}_\sigma$ .*

The extensions, which will also be denoted by  $M_t^+$ ,  $M_t^-$  are unitary on  $\mathcal{H}_\frac{1}{2}$  and, in general, the bound,  $\|M_t^\pm\|_\sigma$  of  $M_t^\pm$  considered as an operator on  $\mathcal{H}_\sigma$  satisfies

$$(6.11) \quad \|M_t^\pm\|_\sigma \leq A_\sigma(1 + |t|)^{\frac{1}{2}}, \quad 0 < \sigma < 1.$$

Since the restrictions of  $m_t^+$ ,  $m_t^-$  to  $L_p$ ,  $1 \leq p < \infty$ , map  $L_p$  isometrically onto  $L_p$ , the Plancherel theorem implies that  $M_t^\pm$  extend to unitary operators on  $\mathcal{H}_1$ .

To treat the case  $\sigma \neq 1/2$ , one would like to express  $M_t^+$  and  $M_t^-$  as integral operators of convolution type. However, the kernels in question are not locally integrable, and we must therefore proceed rather indirectly.<sup>11</sup>

We introduce the transformation

$$m_t^\epsilon: f(x) \rightarrow |x|^{-\epsilon+2it} f(x), \quad 0 < \epsilon < 1/2,$$

which maps  $L_1 \cap L_2$  into  $L_1 \cap L_2$ .

Putting  $F$  for the Fourier transform (6.6) of  $f \in L_1 \cap L_2$ , we denote the Fourier transform of  $m_t^\epsilon(f)$  by  $M_t^\epsilon(F)$ . Now let  $F \in \mathcal{D}$ . Since  $F$  is the Fourier transform of an  $f \in L_1 \cap L_2$ , we can form  $M_t^\epsilon(F)$ , and, as is easily verified, by the Plancherel theorem,

$$(6.12) \quad \|M_t^\epsilon(F) - M_t^+(F)\|_2 \rightarrow 0, \text{ as } \epsilon \rightarrow 0.$$

Next, we claim that

$$(6.13) \quad M_t^\epsilon(F) = a_{\epsilon,t} \int_{-\infty}^{\infty} F(y) |x-y|^{\epsilon-1-2it} dy$$

for  $F$  in  $\mathcal{D}$ , where

$$a_{\epsilon,t} = \frac{1}{\pi} \Gamma(1-\epsilon+2it) \cos[\pi/2(1-\epsilon+2it)].$$

This follows from the fact [2, p. 43] that the Fourier transform of

$$e^{-b|x|} |x|^{-\epsilon+2it}, \quad b > 0$$

$$(6.14) \quad (2\pi)^{-\frac{1}{2}} \Gamma(1-\epsilon+2it) [(b+ix)^{\epsilon-1-2it} + (b-ix)^{\epsilon-1-2it}].$$

This converges to

$$(6.15) \quad (2\pi)^{-\frac{1}{2}} a_{\epsilon,t} |x|^{\epsilon-1-2it}$$

<sup>11</sup> The following observations may help clarify the situation. When  $t=0$ ,  $M_t^+$  reduces to the identity transform. This may be regarded as convolution by the Dirac kernel. When  $t=0$ ,  $M_t^-$  reduces to the so-called "Hilbert transform," which apart from a constant factor may be viewed as a convolution by the function  $1/x$ . In this case our result was proved by Hardy and Littlewood [11], whose argument we extend to the general case.

as  $b \rightarrow 0^+$ , and is bounded by  $A |x|^{\epsilon-1}$ , with  $A$  independent of  $b$ . Now (6.13) follows by standard convergence theorems and the Plancherel theorem.

Together with  $F \in \mathcal{D}$ , consider  $G(x) = |x|^{\sigma-\frac{1}{2}}F(x)$ . Since  $F \in C^\infty$  and vanishes in a neighborhood of zero and outside of a compact set, the same may be said of  $G(x)$ . Thus we may apply formula (6.10) to  $G$  as well. Call

$$(6.16) \quad \Delta_\epsilon(x) = M_t^\epsilon(G)(x) - |x|^{\sigma-\frac{1}{2}}M_t^\epsilon(F)(x).$$

Then by (6.13),

$$(6.17) \quad \Delta_\epsilon(x) = a_{\epsilon,t} \int_{-\infty}^{+\infty} [|y|^{\sigma-\frac{1}{2}}F(y) - |x|^{\sigma-\frac{1}{2}}F(y)] |x-y|^{\epsilon-1-2it} dy.$$

It is easy to verify, (by the Lebesgue dominated convergence theorem) that

$$(6.18) \quad \|\Delta_\epsilon(x) - \Delta_0(x)\|_2 \rightarrow 0, \text{ as } \epsilon \rightarrow 0, (F \in \mathcal{D}).$$

If we use (6.12) with  $G$  in place of  $F$ , (6.18) and (6.16), we obtain

$$(6.19) \quad \| |x|^{\sigma-\frac{1}{2}}M_t^+(F) \|_2 \leq \| M_t^+(G) \|_2 + \| \Delta_0(x) \|_2.$$

As has already been noted

$$\| M_t^+(G) \|_2 = \| G \|_2$$

while

$$\| G \|_2 = \| |x|^{\sigma-\frac{1}{2}}F \|_2 = \| F \|_\sigma,$$

and

$$\| |x|^{\sigma-\frac{1}{2}}M_t^+(F) \|_2 = \| M_t^+(F) \|_\sigma.$$

Substituting the above in (6.19) leads to

$$(6.20) \quad \| M_t^+(F) \|_\sigma \leq \| F \|_\sigma + \| \Delta_0(x) \|_2.$$

It remains therefore to estimate  $\| \Delta_0(x) \|_2$ . Recalling (6.17) we have

$$\begin{aligned} \Delta_0(x) &= a_{0,t} \int_{-\infty}^{+\infty} |y|^{\sigma-\frac{1}{2}}F(y) - |x|^{\sigma-\frac{1}{2}}F(y) |x-y|^{-1-2it} dy \\ &= a_{0,t} \int_{-\infty}^{+\infty} 1 - (|x|/|y|)^{\sigma-\frac{1}{2}} |x-y|^{-1-2it} |y|^{\sigma-\frac{1}{2}}F(y) dy. \end{aligned}$$

We now apply Lemma 14, with  $\alpha = \sigma - \frac{1}{2}$ , and with  $|y|^{\sigma-\frac{1}{2}}F(y)$  in place of  $F(y)$ . We then have

$$\| \Delta_0(x) \|_2 \leq A_\sigma |a_{0,t}| \| |x|^{\sigma-\frac{1}{2}}F(x) \|_2 = A_\sigma |a_{0,t}| \| F(x) \|_\sigma.$$

However

$$a_{0,t} = (1/\pi)\Gamma(1+2it)\cos \frac{1}{2}\pi(1+2it).$$

Hence by well-known estimates in the theory of the  $\Gamma$  function, see [22], p. 151, it follows that

$$|a_{0,t}| \leq A(1 + |t|)^{\frac{1}{2}}.$$

Combining this with the above we obtain:

$$\|\Delta_0(x)\|_2 \leq A_\sigma(1 + |t|)^{\frac{1}{2}} \|F\|_\sigma.$$

Together with (6.20), this implies

$$\|M_t^+(F)\|_\sigma \leq A_\sigma(1 + |t|)^{\frac{1}{2}} \|F\|_\sigma.$$

This was our desired result for  $M_t^+$ .

The proof for  $M_t^-$  is very similar. The only change that occurs is that we use the fact that the Fourier transform of  $\operatorname{sgn}(x) |x|^{-\epsilon+2it}$  is

$$(2\pi)^{\frac{1}{2}} b_{\epsilon,t} \operatorname{sgn}(x) |x|^{\epsilon-1-2it},$$

where

$$b_{\epsilon,t} = (i/\pi) \Gamma(1 - \epsilon + 2it) \sin \frac{1}{2}\pi [1 - \epsilon + 2it].$$

This concludes the proof of the lemma.

LEMMA 16. *The estimates for  $M_t^+$  and  $M_t^-$  may be strengthened as follows. Let  $\epsilon > 0$ , then*

$$\|M_t^+\|_\sigma \leq A_{\sigma,\epsilon}(1 + |t|)^{(1+\epsilon)|\sigma-\frac{1}{2}|}, \quad 0 < \sigma < 1,$$

with  $A_{\sigma,\epsilon}$  independent of  $t$ .

*Proof.* Let us consider  $M_t^+$ , and assume that  $\frac{1}{2} \leq \sigma < 1$ ; the other cases are treated analogously. We have already noted that  $M_t^+$  is unitary on  $\mathcal{H}_1$ . Thus we have

$$(6.21) \quad \left( \int_{-\infty}^{+\infty} |M_t^+(F)|^2 dx \right)^{\frac{1}{2}} = \left( \int_{-\infty}^{+\infty} |F|^2 dx \right)^{\frac{1}{2}}.$$

By the lemma we have just proved, we have, if  $\frac{1}{2} \leq \sigma_0 < 1$ ,

$$(6.22) \quad \begin{aligned} & \left( \int_{-\infty}^{+\infty} |M_t^+(F)|^2 |x|^{2\sigma_0-1} dx \right)^{\frac{1}{2}} \\ & \leq A_{\sigma_0}(1 + |t|)^{\frac{1}{2}} \left( \int_{-\infty}^{+\infty} |F(x)|^2 |x|^{2\sigma_0-1} dx \right)^{\frac{1}{2}}. \end{aligned}$$

Notice that the above inequalities are of the same nature, except for the weight functions which determine the measures in question.

Now it is possible to "interpolate" between these two inequalities, and obtain intermediate ones from them. Of course we have already used many

variants of this type of argument in § 3 and § 4 above. The particular theorem we need is contained in [20], (Theorem 2). To apply it we argue as follows:

Choose  $\sigma_0$ , so that  $\sigma < \sigma_0 < 1$ . We may then write

$$2\sigma - 1 = (1 - \theta) \cdot 0 + \theta(2\sigma_0 - 1) = \theta(2\sigma_0 - 1),$$

with  $0 < \theta < 1$ . Notice that in the above,  $\sigma = \frac{1}{2}$  when  $\theta = 0$ , and  $\sigma = \sigma_0$  when  $\theta = 1$ . The result of applying Theorem 2 of [20] is

$$\begin{aligned} & \left( \int_{-\infty}^{+\infty} |M_t^+(F)|^2 |x|^{2\sigma-1} dx \right)^{\frac{1}{2}} \\ & \leq A_{\sigma_0} \theta (1 + |t|)^{\theta/2} \left( \int_{-\infty}^{+\infty} |F(x)|^2 |x|^{2\sigma_0-1} dx \right)^{\frac{1}{2}}. \end{aligned}$$

However,

$$\theta = (2\sigma - 1)/(2\sigma_0 - 1).$$

Thus we choose  $\sigma_0$  close enough to 1 so that  $\theta \leq (2\sigma - 1)(1 + \epsilon)$ . Hence the result becomes

$$\begin{aligned} & \left( \int_{-\infty}^{+\infty} |M_t^+(F)|^2 |x|^{2\sigma-1} dx \right)^{\frac{1}{2}} \\ & \leq A_{\sigma, \epsilon} (1 + |t|)^{(\sigma-1/2)(1+\epsilon)} \left( \int_{-\infty}^{+\infty} |F(x)|^2 |x|^{2\sigma_0-1} dx \right)^{\frac{1}{2}}. \end{aligned}$$

Our lemma is therefore proved.

*Remark.* We observe that the above proof yields the inequality

$$A_{\sigma, \epsilon} \leq A_{\sigma_0} \theta.$$

A simple argument then allows us to deduce the following fact: The constant  $A_\sigma$  which appears in (6.11) may be taken to be uniformly bounded in every closed subinterval of  $\sigma$  lying in  $0 < \sigma < 1$ .

This observation will be of use later.

## CHAPTER II. UNIFORMLY BOUNDED REPRESENTATIONS.

**7. Proofs of Theorem 5 and Theorem 6.** Before presenting the details of the argument, we shall briefly discuss the main steps involved in the construction of the representations  $g \rightarrow U^z(g, s)$ .

Our representations are constructed on the space  $\mathcal{H} = \mathcal{H}_1$  from representations  $g \rightarrow V^z(g, s)$  on  $\mathcal{H}_\sigma$ ,<sup>12</sup>  $s = \sigma + it$ . The operators  $U^z(g, s)$  and  $V^z(g, s)$  are related by

<sup>12</sup> For the definition of the Hilbert space  $\mathcal{H}_\sigma$  see (6.1).

$$(7.1) \quad U^z(g, s) = W(s, \tfrac{1}{2}) V^z(g, s) W(\tfrac{1}{2}, s),$$

where  $W(s, \tfrac{1}{2})$  is the unitary transformation (6.2) of  $\mathcal{H}_\sigma$  onto  $\mathcal{H}_\frac{1}{2}$ . The representations  $g \rightarrow V^z(g, \tfrac{1}{2} + it)$  are obtained by simply transferring the representations  $g \rightarrow v^z(g, \tfrac{1}{2} + it)$  of the continuous principal series from  $L_2$  to  $\mathcal{H}_\frac{1}{2}$ , by means of the Fourier transform. We also obtain the operators  $V^z(g, s)$ ,  $0 < Rs < \tfrac{1}{2}$ , via the Fourier transform in a similar, but technically more involved, fashion from the representations  $g \rightarrow v^z(g, s)$ . To define the operators  $V^z(g, s)$  for  $\tfrac{1}{2} < Rs < 1$  it is convenient to extend the notation  $\sigma' = 1 - \sigma$  to complex  $s$  with  $0 < Rs < 1$  by setting  $s' = 1 - \bar{s}$ ; the transformation  $s \rightarrow s'$  is then simply reflection about the line  $\sigma = \tfrac{1}{2}$ . Now the representation corresponding to an  $s$  with  $\tfrac{1}{2} < Rs < 1$  is defined to be the contragredient of the representation corresponding to  $s'$ . Thus we put<sup>13</sup>

$$(7.2) \quad V^z(g, s) = [V^z(g^{-1}, s')]', \quad \tfrac{1}{2} < Rs < 1.$$

It follows that

$$(7.3) \quad [V^z(g^{-1}, s)]' = V^z(g, s'), \quad 0 < Rs < 1.$$

It will be shown in the course of the proof that the apparently arbitrary definition (7.2) is the natural one to make.

As a first step in the proof we shall establish the following lemma.

LEMMA 17. *The multipliers  $\phi^z$  given by (5.2), (5.3) satisfy*

- a)  $\phi^z(g, x, s) = \overline{\phi^z(g, x, \bar{s})}$ ,
- b)  $\phi^z(g_1 g_2, x, s) = \phi^z(g_1, x, s) \phi^z(g_2, g_1 x, s)$ ,
- c)  $\phi^z(g, g^{-1}x, s) dg^{-1}(x)/dx = \phi^z(g^{-1}, x, 1 - s)$ .

The first relation, a) is immediate, b) is essentially a consequence of the chain rule for derivatives applied to (5.1), and c) follows by simple computations from b) upon setting  $g_2 = g$  and  $g_1 = g^{-1}$ .

As the following lemma shows, it is natural to restrict  $s$  so that  $0 \leq Rx < 1$ .

LEMMA 18. *Suppose  $s = \sigma + it$ , where  $0 \leq \sigma \leq 1$ . Then for each  $g \in G$ , the operators  $v^z(g, s)$  are isometric on  $L_p$ , where  $p = (1 - \sigma)^{-1}$ .*

Since the case  $p = \infty$  is easily verified, we shall suppose  $1 \leq p < \infty$ . Making the transformation  $x \rightarrow g(x)$  we find that

<sup>13</sup> If  $A$  is an operator on  $\mathcal{H}_\sigma$ ,  $A'$  is the operator on  $\mathcal{H}_{\sigma'}$  given by (6.6) and (6.7).

$$\int_{-\infty}^{\infty} |f(x)|^p dx = \int_{-\infty}^{\infty} |bx + d|^{-2} |f(g(x))|^p dx.$$

Now  $|v^*(g, s)f(x)|^p = |bx + d|^{(2\sigma-2)p} |f(g(x))|^p$ , and since  $(2\sigma-2)p = -2$ , it follows that  $\|v^*(g, s)f\|_p = \|f\|_p$ .

It is interesting to observe that when  $p = (1-\sigma)^{-1}$  its conjugate index  $p'$  is given by  $p' = (1-\sigma')^{-1}$ . Thus the operators  $v^*(g, s)$  and  $v^*(g, s')$  give rise to a pair of isometric representations of  $G$  on  $L_p, L_{p'}$  where  $p = (1-\sigma)^{-1}$  and  $s = \sigma + it$ . Moreover, as the following lemma shows, these representations are contragredient.

LEMMA 19. *Let  $s = \sigma + it$  and  $p = (1-\sigma)^{-1}$ ,  $0 < \sigma < 1$ . Then for any  $g \in G$ ,  $f \in L_p$ , and  $h \in L_{p'}$ ,*

$$(7.4) \quad (v^*(g, s)f, h) = (f, v^*(g^{-1}, s')h).$$

To prove this we make the transformation  $x \rightarrow g^{-1}(x)$  and find that

$$\begin{aligned} (v^*(g, s)f, h) &= \int_{-\infty}^{\infty} \phi^*(g, x, s) f(g(x)) \overline{h(x)} dx \\ &= \int_{-\infty}^{\infty} \phi^*(g, g^{-1}x, s) f(x) \overline{h(g^{-1}x)} (dg^{-1}(x)/dx) dx. \end{aligned}$$

Thus, by c) of Lemma 17,

$$(v^*(g, s)f, h) = \int_{-\infty}^{\infty} f(x) \phi^*(g^{-1}, x, 1-s) \overline{h(g^{-1}x)} dx,$$

and now part a) of the same lemma shows that

$$(v^*(g, s)f, h) = (f, v^*(g^{-1}, s')h).$$

We now consider the representation spaces  $H_\sigma$  of the complementary series. These spaces are described in the following lemma.<sup>14</sup>

LEMMA 20. *Let  $0 < \sigma < \frac{1}{2}$  and  $p = (1-\sigma)^{-1}$ . Then the inner product (5.5) is well defined for  $f$  in  $L_p$ , and the completion  $H_\sigma$  of  $L_p$  with respect to the norm  $\|f\|_\sigma^2 = (f, f)_\sigma$  is unitarily equivalent to  $\mathcal{H}_\sigma$  via a mapping which coincides with the Fourier transform on  $L_p$ .*

To prove this, suppose first that  $f \in L_1 \cap L_2$  and that  $F$  is its Fourier transform. By (6.14), which is valid for  $0 < \epsilon < 1$ , and the dominated convergence theorem we obtain

$$(7.5) \quad \int_{-\infty}^{\infty} |F(x)|^2 |x|^{2\sigma-1} dx = a_\sigma \int_{-\infty}^{\infty} f^* * f(x) |x|^{-2\sigma} dx.$$

<sup>14</sup> Lemmas 20, 21, and 22 are essentially restatements of known facts.

By Lemma 13 the left side of (7.5) is finite for  $f \in L_p$ , and by simple approximation arguments, it follows that the right side of (7.5) exists and equals the left side for all  $f$  in  $L_p$ . This shows that the formula (5.5) defines an inner product on  $L_p$ . Now observe that the Fourier transform of  $L_p$ ,  $1 < p \leq 2$ , includes the characteristics functions of finite intervals and their linear combinations. This observation together with (7.5) establishes the final statement of the lemma and concludes the proof.

As a consequence of Lemma 18 and Lemma 20, we obtain the fact that the representations  $g \rightarrow v^\pm(g, s)$  are defined on a dense linear subset of  $H_\sigma$ ,  $0 < \sigma < \frac{1}{2}$ . Moreover, as the following lemma shows, the operators  $v^\pm(g, \sigma)$  extend uniquely to unitary operators on  $H_\sigma$ .

LEMMA 21. Let  $0 < \sigma < \frac{1}{2}$  and  $p = (1 - \sigma)^{-1}$ . Then for  $f$  in  $L_p$ ,

$$(7.6) \quad \|v^\pm(g, \sigma)f\|_\sigma = \|f\|_\sigma.$$

In proving this, we use the fact that

$$g(x) - g(y) = (x - y)(bx + d)^{-1}(by + d)^{-1},$$

which follows by straightforward computation. Then making the transformations  $x \rightarrow g(x)$  and  $y \rightarrow g(y)$  we see that

$$\begin{aligned} \|f\|_\sigma^2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)\overline{f(y)} |x - y|^{-2\sigma} dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(g(x))\overline{f(g(y))} |x - y|^{-2\sigma} |bx + d|^{2\sigma-2} |by + d|^{2\sigma-2} dx dy \\ &= \|v^\pm(g, \sigma)f\|_\sigma^2. \end{aligned}$$

Next we shall show that there exists a uniform bound independent of  $g$  for the operators  $v^\pm(g, s)$  in  $H_\sigma$ ;  $s = \sigma + it$ ,  $0 < \sigma < \frac{1}{2}$ . In doing this, we consider the lower triangular subgroup of  $G$  consisting of elements  $g$  of the form

$$g = \begin{bmatrix} a^{-1} & 0 \\ c & a \end{bmatrix}, \quad a \neq 0.$$

We make essential use of the fact that there are only two distinct double cosets of  $G$  modulo this subgroup. To be explicit, we introduce the group element

$$(7.7) \quad j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and prove the following result.

LEMMA 22. If  $g \in G$  and is not lower triangular, there exist lower triangular group elements  $g_1$  and  $g_2$  such that

$$(7.8) \quad g = g_1 j g_2.$$

We prove this by exhibiting such a decomposition. If  $g \in G$  and is not lower triangular we may write

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad b \neq 0.$$

Then as is easily verified

$$g = \begin{bmatrix} 1 & 0 \\ db^{-1} & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} b^{-1} & 0 \\ a & b \end{bmatrix}.$$

In view of this result and the fact that  $v^*(g_1 g_2, s) = v^*(g_1, s) v^*(g_2, s)$  for all  $g_1, g_2$  in  $G$ , it is natural to consider the operators,  $v^*(g, s)$ , first for  $g$  in the lower triangular subgroup and then for  $g = j$ .

LEMMA 23. Let  $s = \sigma + it$ , where  $0 < \sigma < \frac{1}{2}$  and  $-\infty < t < \infty$ . Then

1) if  $g$  is lower triangular,  $v^*(g, s)$  has a unique unitary extension to all of  $H_\sigma$ , and<sup>15</sup>

$$2) \quad \|v^*(j, s)\|_\sigma \leq A_\sigma(1 + |t|)^{\frac{1}{2}}.$$

In proving 1), we suppose that  $g = \begin{bmatrix} a^{-1} & 0 \\ c & a \end{bmatrix}$ ,  $a \neq 0$ . Then by definition

$$(7.9) \quad v^*(g, s) : f(x) \rightarrow |a|^{2s-2} f(a^{-2}x + a^{-1}c),$$

and  $v^-(g, s) = \text{sgn}(a) v^*(g, s)$ . Furthermore,  $v^*(g, s) = |a|^{2it} v^*(g, \sigma)$ ; these relations together with Lemma 21 establish part 1). Turning now to the operators  $v^*(j, s)$  we find that

$$(7.10) \quad v^*(j, s) : f(x) \rightarrow |x|^{2s-2} f(-1/x),$$

$$(7.11) \quad v^-(j, s) : f(x) \rightarrow \text{sgn}(x) |x|^{2s-2} f(-1/x).$$

Now with the aid of the operators  $m_i^+$  and  $m_i^-$  given by (6.9) and (6.10) we can write  $v^*(j, s) = m_i^+ v^*(j, \sigma)$  and  $v^-(j, s) = m_i^- v^*(j, \sigma)$ . Since  $v^*(j, \sigma)$  has a unitary extension, it follows that the bounds of the operators  $v^*(j, s)$  are exactly the bounds of  $m_i^+$ ,  $m_i^-$ , considered as acting in  $H_\sigma$ . Now using the fact that  $H_\sigma$  is unitarily equivalent to  $\mathfrak{H}_\sigma$  and the definitions of  $M_i^+$ ,  $M_i^-$  we obtain 2) as a consequence of Lemma 15.

<sup>15</sup> The symbol  $\|\cdot\|_\sigma$  designates the bound of the operator on  $\mathfrak{H}_\sigma$ .

Finally, using Lemma 22 and Lemma 23, we find that

$$(7.12) \quad \sup_g \|v^\pm(g, s)\|_\sigma \leq A_\sigma(1 + |t|)^{\frac{1}{2}}, \quad 0 < \sigma < \frac{1}{2}.$$

Because of (7.12) we may, and shall from now on, assume that the operators  $v^\pm(g, s)$  are everywhere defined on  $H_\sigma$ .

Since  $H_\sigma$  and  $\mathcal{H}_\sigma$  are unitarily equivalent we may transfer the representations  $g \rightarrow v^\pm(g, s)$  to  $\mathcal{H}_\sigma$  and obtain equivalent representations  $g \rightarrow V^\pm(g, s)$ . The operators  $V^\pm(g, s)$  are obtained as follows: Let  $\mathcal{F}_\sigma$ ,  $0 < \sigma < \frac{1}{2}$ , be the unitary transformation from  $H_\sigma$  to  $\mathcal{H}_\sigma$  that coincides with the Fourier transform (6.6) on  $L_p$ ,  $p = (1 - \sigma)^{-1}$ . In addition let  $\mathcal{F}_2$  be the Fourier transform restricted to  $L_2$ ; we note that  $\mathcal{F}_2$  is unitary between  $L_2$  and  $\mathcal{H}_1$ . We now define  $V^\pm(g, s)$  for  $s = \sigma + it$  by

$$(7.13) \quad V^\pm(g, s) = \mathcal{F}_\sigma v^\pm(g, s) \mathcal{F}_\sigma^{-1}, \quad 0 < \sigma \leq \frac{1}{2}.$$

From (7.12) and the definitions (7.13), (7.2) we obtain the bounds

$$(7.14) \quad \sup_g \|V^\pm(g, s)\|_\sigma \leq A_\sigma(1 + |t|)^{\frac{1}{2}}, \quad 0 < \sigma < 1.$$

This result together with (7.1) implies

$$(7.15) \quad \sup_g \|U^\pm(g, s)\|_\infty \leq A_\sigma(1 + |t|)^{\frac{1}{2}}, \quad 0 < \sigma < 1.$$

Moreover, as the remark at the end of §6 states, we may assume  $A_\sigma$  is bounded on any closed subinterval of  $(0, 1)$ . Hence we have proved 5) of Theorem 5, and conclusions 2) and 3) follow from (7.13).

To show that (7.2) is a natural definition we consider once again the class of functions  $\mathcal{D}$  introduced in §6. Recall that  $F \in \mathcal{D}$  if  $F$  is  $C^\infty$  and vanishes in a neighborhood of zero and outside a compact set.

**LEMMA 24.** *Suppose  $F, H \in \mathcal{D}$  and that  $f, h$  are their Fourier transforms. Then for all  $s$  in the strip  $0 < Rs < 1$ ,*

$$(7.16) \quad (v^\pm(g, s)f, h) = (V^\pm(g, s)F, H).$$

To prove this we suppose first of all that  $0 < Rs \leq \frac{1}{2}$ . Then  $f, v^\pm(g, s)f \in L_p$ ,  $p = (1 - \sigma)^{-1}$ , and  $1 < p \leq 2$ . Our result, (7.6), now follows from the definition of  $V^\pm(g, s)F$  and the Parseval formula for  $L_p, L_{p'}$ , which is stated in Lemma 12. In case  $\frac{1}{2} < Rs < 1$ ,  $V^\pm(g, s) = [V^\pm(g^{-1}, s')]'$ . Thus

$$(V^\pm(g, s)F, H) = (F, V^\pm(g^{-1}, s')H).$$

By the result just established,

$$(F, V^\pm(g^{-1}, s')H) = (f, v^\pm(g^{-1}, s')h).$$

Now applying Lemma 19, we see that

$$(f, v^+(g^{-1}, s')h) = (v^+(g, s)f, h).$$

Thus (7.16) also holds for  $\frac{1}{2} < Rs < 1$ , and hence for all  $s$  in  $0 < Rs < 1$ .

To prove that the representations  $g \rightarrow U^+(g, s)$ , defined by (7.1), are continuous, it suffices to prove that the representations  $g \rightarrow V^+(g, s)$  are; and for this, it is sufficient by (7.2) to consider the case  $0 < Rs \leq \frac{1}{2}$ . Now if  $f$  is continuous and has compact support, it may be shown that for bounded functions  $h$ ,

$$\int_{-\infty}^{\infty} \phi^+(g, x, s) f(g(x)) h(x) dx \rightarrow \int_{-\infty}^{\infty} f(x) h(x) dx$$

as  $g \rightarrow e$ ,  $e$  being the identity in  $G$ . Because the representations  $g \rightarrow v^+(g, s)$ ,  $0 < Rs \leq \frac{1}{2}$ , are uniformly bounded on  $H_\sigma$  this is sufficient to insure their continuity. Hence the equivalent representations  $g \rightarrow V^+(g, s)$  are also continuous.

It remains to prove conclusion 4) which refers to the analyticity of the operators  $U^+(g, s)$ . For this purpose we prove a result which has some interest in its own right.

LEMMA 25. *If  $g$  is a lower triangular matrix in  $G$  the operators  $U^+(g, s)$  are independent of  $s$ ,  $0 < Rs < 1$ .*

Let

$$g = \begin{bmatrix} a^{-1} & 0 \\ c & a \end{bmatrix}, \quad a \neq 0,$$

and choose  $F \in \mathcal{H}_\sigma$ . It then follows from (7.9) and well known properties of the Fourier transform that

$$(7.17) \quad V^+(g, s) : F(x) \rightarrow e^{ixac} |a|^{2s} F(a^2x).$$

We also obtain the relation  $V^-(g, s) = \text{sgn}(a) V^+(g, s)$ . Starting now with  $F \in \mathcal{H}$ , we have by definition that

$$U^+(g, s)F = W(s, \frac{1}{2}) V^+(g, s) W(\frac{1}{2}, s) F.$$

Hence by (7.17),

$$V^+(g, s) W(\frac{1}{2}, s) : F(x) \rightarrow e^{ixac} |a|^{2s} |a^2x|^{\frac{1}{2}-s} F(a^2x)$$

and applying  $W(s, \frac{1}{2})$  we get

$$(7.18) \quad U^+(g, s) : F(x) \rightarrow e^{ixac} |a| F(a^2x).$$

Similarly, we obtain the relation  $U^-(g, s) = \text{sgn}(a)U^+(g, s)$ . Thus we have proved the lemma.

This result shows that the inner products  $(U^\pm(g, s)\xi, \eta)$  are constant as functions of  $s$ , and hence analytic, for any fixed lower triangular  $g \in G$ . Now if  $g$  is not lower triangular, it has a decomposition  $g = g_1 j g_2$  of the type (7.8). Since

$$U^\pm(g, s) = U^\pm(g_1, s)U^\pm(j, s)U^\pm(g_2, s),$$

where  $U^\pm(g_i, s)$ ,  $i = 1, 2$ , are independent of  $s$  and have bounded inverses, it is sufficient to show that  $(U^\pm(j, s)\xi, \eta)$  is analytic in  $s$  for each pair of vectors  $\xi, \eta$  in  $\mathcal{H}$ . Recall the uniform bound, (7.15) for the representations  $g \rightarrow U^\pm(g, s)$ . Since the constant  $A_\sigma$  which appears is bounded as a function of  $\sigma$  over any closed subinterval of  $(0, 1)$ , it is sufficient to prove that  $(U^\pm(j, s)\xi, \eta)$  is analytic for a dense collection of vectors in  $\mathcal{H}$ . Choose this collection to be the set  $\mathcal{D}$  of functions which are  $C^\infty$  and vanish in a neighborhood of zero, and outside a compact set. Pick  $\xi = F$  and  $\eta = H$  in  $\mathcal{D}$ . Let  $F_s(x) = |x|^{\frac{1}{2}-s}F(x)$  and put

$$H_s(x) = |x|^{\sigma-\frac{1}{2}-s}H(x).$$

It is then easily verified that

$$(U^\pm(j, s)\xi, \eta) = (V^\pm(j, s)F_s, H_s).$$

Denote the Fourier transforms of  $F_s, H_s$  by  $f_s, h_s$ . Then as  $F_s, H_s$  belong to  $\mathcal{D}$ , Lemma 24 applies, and we see that

$$(U^\pm(j, s)\xi, \eta) = (v^\pm(j, s)f_s, h_s).$$

Now using (7.10), (7.11) we obtain

$$(7.19) \quad (U^+(j, s)\xi, \eta) = \int_{-\infty}^{\infty} |x|^{2s-2} f_s(-1/x) \overline{h_s(x)} dx.$$

$$(7.20) \quad (U^-(j, s)\xi, \eta) = \int_{-\infty}^{\infty} \text{sgn}(x) |x|^{2s-2} f_s(-1/x) \overline{h_s(x)} dx.$$

Since

$$f_s(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{ixy} |y|^{\frac{1}{2}-s} F(y) dy,$$

and in view of the various restriction on  $F$ , we may conclude that  $f_s(x)$  has the following properties: it is jointly continuous as a function of  $x$  and  $s$ ; it is analytic in  $s$  for each fixed  $x$ ; and if  $s$  is restricted to any compact subset of the strip,  $0 < \text{Re } s < 1$ ,  $f_s(x)$  decreases as  $|x| \rightarrow \infty$  as fast as any negative power of  $|x|$ . Since

$$h_s(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-ixy} |y|^{s-1} \overline{H(y)} dy$$

it has the same properties. It is now a very straightforward matter that (7.19), (7.20) can be obtained as uniform limits of functions analytic in  $s$ . Hence the inner products  $(U^z(j, s)\xi, \eta)$  are analytic in  $s$ . This concludes the proof of Theorem 5.

**COROLLARY.** *Conclusion 5) of Theorem 5 may be strengthened as follows. Given any  $\epsilon > 0$ , then*

$$\sup_g \|U^z(g, s)\|_{\infty} \leq A_{\sigma, \epsilon} (1 + |t|)^{|\sigma-1|(1+\epsilon)}$$

for  $s = \sigma + it$ ,  $0 < \sigma < 1$ .

In proving the theorem we made use of the estimate given by (6.11). If, however, we had used the estimate given by Lemma 16, we would have obtained the above.

We shall now Prove 1) of Theorem 6, which asserts that the representations  $U^z(\cdot, s)$  and  $U^z(\cdot, s')$  are contragredient.

In order to do this, we first combine (7.3) and (6.6) to obtain

$$(7.21) \quad V^z(g, s') = W_{\sigma} V^z(g^{-1}, s) * W_{\sigma}^{-1}.$$

It then follows by definition that

$$U^z(g, s') = W(s', \frac{1}{2}) W_{\sigma} V^z(g^{-1}, s) * W_{\sigma}^{-1} W(\frac{1}{2}, s').$$

Using the definitions of  $W(s', \frac{1}{2})$ ,  $W_{\sigma}$  together with the fact that  $s' - \frac{1}{2} + \sigma - \sigma' = s - \frac{1}{2}$  we find that

$$W(s', \frac{1}{2}) W_{\sigma} = W(s, \frac{1}{2}).$$

Substituting into the above we obtain

$$U^z(g, s') = W(s, \frac{1}{2}) V^z(g^{-1}, s) * W(\frac{1}{2}, s),$$

which implies,

$$(7.22) \quad U^z(g, s') = U^z(g^{-1}, s)^*.$$

Hence we have proved part 1).

The second statement of Theorem 6 is easily seen to follow from the fact that the representations  $g \rightarrow U^*(g, \sigma)$  are unitary for  $0 < \sigma < 1$ .

In fact, suppose that  $g \rightarrow U(g, s)$  are any representations of  $G$  on  $\mathcal{H}$  such that

$$U(g, s') = U(g^{-1}, s)^*$$

and for which the inner products  $(U(g, s)\xi, \eta)$  are analytic in  $s$ . Then

$$U(\cdot, s) = U(\cdot, 1 - s)$$

if and only if the representations  $U(\cdot, \sigma)$  are unitary for each  $\sigma$ ,  $0 < \sigma < 1$ .

To prove this, we observe that the condition  $U(\cdot, s) = U(\cdot, 1 - s)$  is, by analyticity, equivalent to the condition  $U(\cdot, \sigma) = U(\cdot, 1 - \sigma)$  for  $0 < \sigma < 1$ . On the other hand,  $1 - \sigma = \sigma'$  so that  $U(g^{-1}, \sigma)^* = U(g, 1 - \sigma)$ . Hence the above is equivalent to the condition  $U(g, \sigma) = U(g^{-1}, \sigma)^*$ ,  $g \in G$ .

It is interesting to note that the representations  $U(\cdot, s)$  do not satisfy 2). This is a reflection of the known fact that they are not unitary when  $\sigma \neq \frac{1}{2}$ . In order to prove this, it is sufficient to show that the representations  $v(\cdot, \sigma)$ ,  $0 < \sigma < \frac{1}{2}$ , are not unitary. Without going into detail, we remark that this is a consequence of the relation

$$(7.23) \quad \begin{aligned} & \|v(j, \sigma)f\|_{\sigma^2}^2 - \|f\|_{\sigma^2}^2 \\ &= a_{\sigma} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\operatorname{sgn}(x)\operatorname{sgn}(y) - 1)f(x)f(y)|x - y|^{-2\sigma} dx dy, \end{aligned}$$

which is valid for all  $f$  in  $L_p$ ,  $p = (1 - \sigma)^{-1}$ .

We suppose now that  $S$  is a bounded operator with a bounded inverse such that

$$SU^-(\cdot, s)S^{-1} = U^-(\cdot, 1 - s).$$

Replacing  $s$  by  $1 - s$  and making simple calculations we find that

$$S^2U^-(\cdot, s) = U^-(\cdot, s)S^2.$$

We shall assume the known fact that the unitary representations  $U^-(\cdot, \frac{1}{2} + it)$ ,  $t \neq 0$ , are irreducible. It then follows that  $S^2$  is a scalar multiple of the identity. For lower triangular group elements  $g$  of the form

$$g = \begin{bmatrix} a^{-1} & 0 \\ c & a \end{bmatrix}, \quad a \neq 0,$$

we know that

$$U^-(g, s) : F(x) \rightarrow e^{ixac} aF(a^2x)$$

and

$$SU^-(g, s) = U^-(g, s)S.$$

Setting  $a = 1$  and then setting  $c = 0$  we find that  $S$  is the operation of multiplicity by a function, say  $K$ , with the property that  $K(x) = K(a^2x)$ . Since  $S^2$  is a scalar multiple of the identity, we obtain the additional relation  $(K(x))^2 = \text{const.}$ , which implies  $K(x) = \text{const.}$  or  $K(x) = (\text{const.})\operatorname{sgn}(x)$ .

As the first alternative holds if and only if  $U^-(\cdot, \sigma)$  is unitary for  $0 < \sigma < 1$  we conclude that  $K(x) = (\text{const.}) \text{sgn}(x)$ .

We shall now define  $S$  by

$$(7.24) \quad S: F(x) \rightarrow \text{sgn}(x)F(x),$$

and prove that  $SU^-(g, s)S^{-1} = U^-(g, 1-s)$  for all  $g$  in  $G$ . This may be shown directly for all  $g$ ; however, such a proof does not exhibit the crux of the matter, which, as it turns out, is the relation

$$SU^-(j, \sigma)S^{-1} = U^-(j, 1-\sigma).$$

We therefore proceed along different lines and first of all recall that the operators  $U^-(g, s)$  are independent of  $s$  for lower triangular group elements  $g$ . For such  $g$ , the above relation becomes

$$SU^-(g, s) = U^-(g, s)S.$$

To verify this, suppose

$$g = \begin{bmatrix} a^{-1} & 0 \\ c & a \end{bmatrix}, \quad a \neq 0.$$

Then

$$SU^-(g, s): F(x) \rightarrow \text{sgn}(x)e^{ixac}aF(a^2x)$$

and

$$U^-(g, s)S: F(x) \rightarrow e^{ixac}a \text{sgn}(a^2x)F(a^2x).$$

In view of the decomposition (7.8) it is therefore seen to be sufficient to prove the relation

$$SU^-(j, s) = U^-(j, 1-s)S;$$

Moreover, by analyticity, it is sufficient to prove this for  $s = \sigma$ ,  $0 < \sigma < \frac{1}{2}$ .

Now

$$SU^-(j, \sigma)S^{-1} = U^-(j, \sigma')$$

if and only if

$$W(\tfrac{1}{2}, \sigma')SU^-(j, \sigma)S^{-1}W(\sigma', \tfrac{1}{2}) = W(\tfrac{1}{2}, \sigma')U^-(j, \sigma')W(\sigma', \tfrac{1}{2}).$$

Thus using the fact that  $S$  commutes with  $W(\tfrac{1}{2}, \sigma')$  we see that it is sufficient to prove

$$(7.25) \quad V^-(j, \sigma') = SW(\sigma, \sigma')V^-(j, \sigma)W(\sigma', \sigma)S^{-1}, \quad 0 < \sigma < \tfrac{1}{2}.$$

In proving (7.25) we use the following considerations. The operation  $SW(\sigma, \sigma')$  is multiplication by  $\text{sgn}(x)|x|^{2\sigma-1}$ . Going over to the Fourier transform, this corresponds to convolution by  $b_\sigma/(2\pi)^{\frac{1}{2}}\text{sgn}(x)|x|^{-2\sigma}$ , where

$$b_{\sigma} = i/\pi \Gamma(2\sigma) \sin \pi \sigma.$$

(This fact may be established in the same way as (6.13) was; for further discussion see the proof of Lemma 15, § 6.)

Recalling the definition of  $V^-(j, \sigma)$  in terms of the Fourier transform, it then suffices to prove the following: the operation of convolution by  $b_{\sigma}/(2\pi)^{\frac{1}{2}} \operatorname{sgn}(x) |x|^{-2\sigma}$ , followed by the operation  $f(x) \rightarrow \operatorname{sgn}(x) |x|^{-2\sigma} f(-1/x)$  is equal to the operation  $f(x) \rightarrow \operatorname{sgn}(x) |x|^{2\sigma-2} f(-1/x)$  followed by convolution with  $b_{\sigma}/(2\pi)^{\frac{1}{2}} \operatorname{sgn}(x) |x|^{-2\sigma}$ . This leads to the verification

$$\begin{aligned} \int_{-\infty}^{\infty} \operatorname{sgn}(x) \operatorname{sgn}(-1/x - y) |x|^{-2\sigma} |-1/x - y|^{-2\sigma} f(y) dy \\ = \int_{-\infty}^{\infty} \operatorname{sgn}(y) \operatorname{sgn}(y - x) |y|^{2\sigma-2} |x - y|^{-2\sigma} f(-1/y) dy. \end{aligned}$$

That this holds may be checked by the obvious change of variables.

The argument above needs to be made precise. We therefore argue as follows.

In proving (7.25) it clearly suffices to show that

$$(V^-(j, \sigma') SW(\sigma, \sigma') F, H) = (SW(\sigma, \sigma') V^-(j, \sigma) (F), H)$$

for  $F, H \in \mathcal{D}$ . Let  $f$  be the Fourier transform of  $F$ . Put

$$f_1(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{ixy} \operatorname{sgn}(y) |y|^{2\sigma-1} F(y) dy.$$

Then by what has been said before,

$$f_1(x) = b_{\sigma}/(2\pi)^{\frac{1}{2}} \int_{-\infty}^{\infty} \operatorname{sgn}(x - y) |x - y|^{-2\sigma} f(y) dy.$$

We define  $h$ , and  $h_1$  similarly; thus it follows that

$$h_1(x) = b_{\sigma}/(2\pi)^{\frac{1}{2}} \int_{-\infty}^{\infty} \operatorname{sgn}(x - y) |x - y|^{-2\sigma} h(y) dy.$$

Since

$$(SW(\sigma, \sigma') V^-(j, \sigma) (F), H) = (V^-(j, \sigma) (F), SW(\sigma, \sigma') H),$$

it suffices in view of Lemma 24 to show that

$$(7.26) \quad (v^-(j, \sigma') f_1, h) = (v^-(j, \sigma) f, h_1).$$

Now,

$$\begin{aligned} (v^-(j, \sigma) f, h_1) &= \int_{-\infty}^{\infty} \operatorname{sgn}(y) |y|^{2\sigma-2} f(-1/y) \overline{h_1(y)} dy \\ &= -b_{\sigma}/(2\pi)^{\frac{1}{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{h(x)} f(-1/y) \operatorname{sgn}(y) \operatorname{sgn}(y - x) |y|^{2\sigma-2} |x - y|^{-2\sigma} dy dx. \end{aligned}$$

On the other hand,

$$(v^-(j, \sigma')f_1, h) = b_\sigma / (2\pi)^{\frac{1}{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x)f(y)\operatorname{sgn}(x)\operatorname{sgn}(-1/x-y)|x|^{-2\sigma}|x-y|^{-2\sigma}dydx.$$

If we make the change of variables  $y \rightarrow -1/y$  in the first double-integral, then it is easily verified that this first double-integral equals the second double integral. This proves (7.26) and concludes the proof of Theorem 6.

### CHAPTER III. THE FOURIER-LAPLACE TRANSFORM ON THE GROUP.

**8. Hausdorff-Young theorem for the group and certain of its implications.** Let  $f \in L_1(G)$ , and let us define the Fourier transform of  $f$  on  $G$  as follows:

$$(8.1) \quad \mathcal{F}^z(s) = U^z(f, s) = \int_G U^z(g, s)f(g)dg, \quad 0 < R(s) < 1, \quad f \in L_1(G).$$

$U^z(\cdot, s)$  is the analytic family of representations which act on  $\mathcal{A}$ , and which were studied in §§ 5, 6 and 7. Because for each fixed  $s$ ,  $0 < R(s) < 1$ ,  $U^z(\cdot, s)$  is a uniformly bounded representation, the integral appearing in (8.1) is well defined. Moreover, if  $\xi, \eta \in \mathcal{A}$ , then

$$(\mathcal{F}^z(s)\xi, \eta) = \int_G (U^z(g, s)\xi, \eta)f(g)dg.$$

An application of Fubini's theorem, and the analyticity of  $U^z(\cdot, s)$  shows that

$$\int_G (F^z(s)\xi, \eta)ds = 0,$$

for any closed curve  $C$  in  $0 < R(s) < 1$ .

Thus the Fourier transform  $\mathcal{F}^z(s)$  is not only well-defined when  $f \in L_1(G)$ , and  $0 < R(s) < 1$ , but is also an analytic operator-valued function of  $s$  in that strip.

The results of this section show, in a very precise way, that one may obtain similar results for the Fourier transform of functions in  $L_p(G)$ ,  $1 \leq p < 2$ . These facts are contained in the following theorem together with its corollaries.<sup>10</sup>

**THEOREM 7.** Let  $1 < p < 2$ , and  $q$  be its conjugate index  $1/p + 1/q = 1$ .

<sup>10</sup> The norms  $\|\cdot\|_q$ ,  $1 \leq q \leq \infty$ , are those introduced in § 2. We recall that  $\|\cdot\|_2$  is the "Hilbert-Schmidt" norm while  $\|\cdot\|_\infty$  denotes the operator bound.

There exists a measure  $d\mu_{q,\sigma}(t)$  so that

$$(8.2) \quad \left( \int_{-\infty}^{\infty} \|\mathcal{F}^z(\sigma + it)\|_q^q d\mu_{q,\sigma}(t) \right)^{1/q} \leq \|f\|_p,$$

$f$  simple,  $s = \sigma + it$ , and  $1/q < \sigma < 1/p$ . For the measure  $d\mu_{q,\sigma}(t)$  we have the following estimate: Given any  $\delta > 0$ , then:

$$d\mu_{q,\sigma}(t) \geq A_{q,\sigma,\delta} (1 + |t|)^{1-q|\sigma-\frac{1}{2}|-\delta} dt.$$

COROLLARY 1. For each fixed  $p$ ,  $1 < p < 2$ , there exists a  $\sigma_0$ ,  $1/q < \sigma_0 < \frac{1}{2}$ , so that

$$\left( \int_{-\infty}^{+\infty} \|\mathcal{F}^z(\sigma + it)\|_q^q dt \right)^{1/q} \leq A_{q,\sigma} \|f\|_p$$

whenever  $\sigma_0 < \sigma < 1 - \sigma_0$ , and  $f$  is simple.

COROLLARY 2. For each  $p$ ,  $1 \leq p < 2$ ,

$$\sup_{-\infty < t < \infty} \|\mathcal{F}^z(\frac{1}{2} + it)\|_{\infty} \leq A_p \|f\|_p, \quad f \text{ simple.}$$

COROLLARY 3. For each  $p$ ,  $1 \leq p < 2$ ,  $1/q < R(s) < 1/p$ ,  $s = \sigma + it$ ,

$$\|\mathcal{F}^z(s)\|_{\infty} \leq A_{p,\sigma,t} \|f\|_p, \quad \text{if simple.}$$

COROLLARY 4. The Fourier transform, initially defined for  $f \in L_1 \cap L_p$ , has a unique bounded extension to all of  $L_p(G)$ ,  $1 \leq p < 2$ , with the following property:  $U^z(f, \cdot)$  is for each  $f \in L_p(G)$  analytic in  $s$ , for  $1/q < R(s) < 1/p$ . Moreover, the extension satisfies (8.2) as well as the conditions of Corollaries 1 through 3.

Remarks. A strict analogue of the classical Hausdorff-Young theorem would have been a result like (8.2), but only for  $\sigma = \frac{1}{2}$ . The above results show, however, that the same conclusion holds for a proper strip which contains the line  $\sigma = \frac{1}{2}$  in its interior. This, together with the analyticity of  $\mathcal{F}^z$ , has far-reaching consequences. Once (8.2) has been proved, the results of Corollaries 2, 3, and 4 follow by rather standard "Phragmen-Lindelöf" type arguments.

It is possible to obtain somewhat stronger versions of Corollaries 2 and 3 by replacing the  $\|\cdot\|_{\infty}$  operator norm by the norm  $\|\cdot\|_q$ . Since these latter results do not seem to have any immediate applications, we have not bothered to give their proofs.

A complete Fourier analysis of an arbitrary function (in the class  $L_2(G)$ ) necessitates together with the continuous principal series also the discrete principal series. The discussion of the discrete principal series is much simpler, and is taken up in the next section.

*Proof of Theorem 7.* Let us consider the case  $\mathcal{F}^+$ , that of  $\mathcal{F}^-$  being entirely similar. On account of the corollary to Theorem 5 (see § 7) we may write down the following inequality:

$$(8.3) \quad \sup_{-\infty < t < \infty} (1 + |t|)^{-|\sigma - \frac{1}{2}|(1+\epsilon)} \|\mathcal{F}^+(\sigma + it)\|_{\infty} \leq A_{\sigma, \epsilon} \|f\|_1, \\ 0 < \sigma < 1, \text{ and } \epsilon > 0.$$

This inequality follows from the above quoted corollary and the observation that

$$\|\mathcal{F}^+(\sigma + it)\|_{\infty} \leq \sup_g \|U^+(g, s)\|_{\infty} \|f\|_1.$$

We know that  $U^+(\cdot, \frac{1}{2} + it)$  is unitarily equivalent to the representation  $v^+(\cdot, \frac{1}{2} + it)$  of the continuous principal series. This series, however, is contained in the Plancherel formula (see § 5). Hence we may write down the following inequality:

$$(8.4) \quad \left( \int_{-\infty}^{\infty} \|\mathcal{F}^+(\tfrac{1}{2} + it)\|_2^2 t^2 (1 + |t|)^{-1} dt \right)^{\frac{1}{2}} \leq A \|f\|_2.$$

Here we have used the semi-trivial observation that,

$$t^2 (1 + |t|)^{-1} \leq A t \tanh \pi t, \quad -\infty < t < \infty.$$

We shall apply Theorem 3 to inequalities (8.3) and (8.4) above. We argue as follows. Assume that  $\sigma$  is given and  $1/q < \sigma < 1/p$ . We assume first that  $\sigma < \frac{1}{2}$ . Let  $\alpha$  be a fixed real number with  $0 < \alpha < \sigma < \frac{1}{2}$ , but otherwise arbitrary. Rewrite (8.3) with  $\alpha$  instead of  $\sigma$ . It becomes

$$(8.5) \quad \sup_{-\infty < t < \infty} |1 + |t||^c \|\mathcal{F}^+(\alpha + it)\|_{\infty} \leq A_{\alpha, \epsilon} \|f\|_1 \\ \text{with } c = (\alpha - \tfrac{1}{2})(1 + \epsilon).$$

Our given  $p$ ,  $1 < p < 2$ , determines a parameter  $\tau$ ,  $0 < \tau < 1$ , with

$$1/p = (1 - \tau) + \tau/2 = 1 - \tau/2, \text{ and } 1/q = \tau/2.$$

Now if  $1/q < \sigma < \frac{1}{2}$ , there always exists an  $\alpha$ ,  $0 < \alpha < \sigma$ , so that

$$(8.6) \quad \sigma = \alpha(1 - \tau) + \beta\tau, \quad (\beta = \tfrac{1}{2}).$$

The above relation determines  $\alpha$  uniquely, which  $\alpha$  we now fix. In applying Theorem 3 to (8.4) and (8.5) we make the following further identifications:

$$(8.7) \quad \begin{cases} c = (\alpha - \tfrac{1}{2})(1 + \epsilon) \\ a = 1 \\ b = \tfrac{1}{2}. \end{cases}$$

Now the result of Theorem 3 is

$$(8.8) \quad \left( \int_{-\infty}^{\infty} \| \mathcal{F}^*(\sigma + it) \|_q^q (1 + |t|)^{dq} dt \right)^{1/q} \leq A_{\epsilon, \tau} \| f \|_p$$

whenever  $f$  is simple.

A straightforward calculation leads to

$$(8.9) \quad dq = 1 - \left| \sigma - \frac{1}{2} \right| q (1 + \epsilon), \quad (\epsilon > 0).$$

Now given any  $\delta > 0$ , we can choose an  $\epsilon > 0$ , small enough so that

$$(8.10) \quad dq = 1 - \left| \sigma - \frac{1}{2} \right| q - \delta.$$

Substituting this value of  $dq$  in (8.8) proves (8.2), whenever  $f$  is simple. The consideration of the case  $\frac{1}{2} < \sigma < 1/p$  is carried out in the same manner once one defines

$$\mathcal{F}^+(s) = \mathcal{F}^*(1 - s).$$

The consideration of  $\mathcal{F}^-(s)$  is analogous to  $\mathcal{F}^+(s)$ . This concludes the proof of Theorem 5.

*Proof of Corollary 1.* Consider the quantity  $1 - q \left| \sigma - \frac{1}{2} \right| - \delta$ . This is the exponent that occurs in the measure  $d\mu_{q, \sigma}(t)$ . Recall that  $\delta$  was arbitrary, except  $\delta > 0$ . Notice that if  $q$  is fixed we can make the quantity non-negative by choosing  $\delta$  small enough and  $\sigma$  sufficiently close to  $\frac{1}{2}$ . However  $\sigma$  is also restricted by  $1/q < \sigma < 1/p$ . Thus it is clear that we can realize the conditions of the corollary if we take

$$\sigma_0 = \max(1/q, \frac{1}{2} - 1/q).$$

Hence for this choice of  $\sigma_0$ , the corollary is proved.

The proofs of the other corollaries necessitate the following lemma which is along very classical lines.

**LEMMA 26.** *Let  $\Phi(s)$  be a (numerical-valued) function analytic in an open region which contains the strip*

$$\alpha \leq R(s) \leq \beta, \quad \alpha < \beta.$$

*Suppose that for some  $c \geq 0$ ,*

$$\sup_{\alpha \leq \sigma \leq \beta} |\Phi(\sigma + it)| = O(|t|^c), \text{ as } |t| \rightarrow \infty,$$

*and furthermore, for some  $q, q \geq 1$ ,*

$$\int_{-\infty}^{\infty} |\Phi(\alpha + it)|^q dt \leq 1, \quad \int_{-\infty}^{\infty} |\Phi(\beta + it)|^q dt \leq 1.$$

*Let  $\alpha < \gamma < \beta$ .*

Conclusion:

$$\sup_{-\infty < t < \infty} |\Phi(\gamma + it)| \leq A.$$

$A$  depends on  $\alpha, \beta, \gamma$ , and  $q$ , but does not otherwise depend on  $c$  or  $\Phi$ .

*Proof.* Let  $p$  be the index conjugate to  $q$ ,  $1/p + 1/p = 1$ . Choose  $\phi$  to be a continuous function on  $(-\infty, \infty)$  which vanishes outside a finite interval, and satisfies

$$(8.11) \quad \int_{-\infty}^{\infty} |\phi(t)|^p dt \leq 1,$$

but let  $\phi$  be arbitrary otherwise.

Define  $\Phi_1(\sigma + it)$  by

$$\Phi_1(\sigma + it) = \int_{-\infty}^{\infty} \Phi(\sigma + it + it_1) \phi(t_1) dt_1, \quad \alpha \leq \sigma \leq \beta.$$

Then it is easy to verify that  $\Phi_1(s)$  is analytic in an open region which contains  $\alpha \leq R(s) \leq \beta$ ; that

$$\sup_{\alpha \leq \sigma \leq \beta} |\Phi_1(\sigma + it)| = O(|t|^c), \text{ as } |t| \rightarrow \infty;$$

and in view of the assumptions on  $\Phi$  and (8.11) that

$$\sup_{-\infty < t < \infty} |\Phi_1(\alpha + it)| \leq 1, \text{ and } \sup_{-\infty < t < \infty} |\Phi_1(\beta + it)| \leq 1.$$

We are now in a position to apply the classical Phragmen-Lindelöf principle to  $\Phi_1$ .<sup>17</sup> The conclusion is that  $|\Phi_1|$  is bounded by 1 in the entire strip  $\alpha \leq R(s) \leq \beta$ . In particular,

$$|\Phi_1(\sigma)| \leq 1, \quad \alpha \leq \sigma \leq \beta.$$

Going back to the definition of  $\Phi_1$ , we obtain

$$\left| \int_{-\infty}^{\infty} \Phi(\sigma + it_1) \phi(t_1) dt_1 \right| \leq 1, \quad \alpha \leq \sigma \leq \beta.$$

Considering the arbitrariness of  $\phi$  (except for condition (8.11)) the converse of Hölder's inequality shows:

$$(8.12) \quad \int_{-\infty}^{\infty} |\Phi(\sigma + it)|^q dt \leq 1, \quad \text{if } \alpha \leq \sigma \leq \beta.$$

For functions which are analytic in a strip and satisfy a uniform estimate like (8.12) there is a known variant of Cauchy's integral formula. It is

<sup>17</sup> See e. g. Titchmarsh [22; p. 181].

$$\begin{aligned}
 (8.13) \quad & \Phi(\gamma + it) \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (\Phi(\alpha + it_1)/(\alpha + it_1 - \gamma - it)) dt_1 \\
 &\quad - \frac{1}{2\pi} \int_{-\infty}^{\infty} (\Phi(\beta + it_1)/(\beta + it_1 - \gamma - it)) dt_1, \\
 &\quad \alpha < \gamma < \beta.
 \end{aligned}$$

In Paley and Wiener ([18] pp. 3-5), (8.13) is demonstrated under the assumption corresponding to  $q=2$  in (8.12). However, the proof in the general case,  $q \geq 1$ , is no different.

If one applies Hölder's inequality to each of the integrals in (8.13) one obtains:

$$(8.14) \quad \sup_{-\infty < t < \infty} |\Phi(\gamma + it)| \leq A_{\alpha\beta\gamma q},$$

where

$$\begin{aligned}
 A_{\alpha\beta\gamma q} = \frac{1}{2\pi} [ & (\int_{-\infty}^{\infty} dt / ((\gamma - \alpha)^2 + t^2)^{p/2})^{1/p} \\
 & + (\int_{-\infty}^{\infty} dt / ((\gamma - \beta)^2 + t^2)^{p/2})^{1/p} ]. \\
 & (1/p + 1/q = 1).
 \end{aligned}$$

A simple calculation shows,

$$(8.15) \quad A_{\alpha,\beta,\gamma,q} \leq c[(\gamma - \alpha)^{-1/q} + (\beta - \gamma)^{-1/q}],$$

and with  $c$  some absolute constant. This concludes the proof of the lemma.

*Proof of Corollary 2.* Let us assume for simplicity that  $\|f\|_p = 1$ .

Considering the index  $\sigma_0$  defined in Corollary 1, choose  $\sigma_0 < \sigma_1 < \frac{1}{2}$ , and keep  $\sigma_1$  fixed throughout the rest of this argument. Now by the choice of  $\sigma_1$  (and the normalization imposed on  $f$ ) we have

$$\begin{aligned}
 (8.16) \quad & \int_{-\infty}^{\infty} \|\mathcal{F}^z(\sigma_1 + it)\|_q^q dt \leq A^q, \\
 & \int_{-\infty}^{\infty} \|\mathcal{F}^z(1 - \sigma_1 + it)\|_q^q dt \leq A^q,
 \end{aligned}$$

with  $A$  independent of  $f$ , for some appropriate  $A$ .

Choose  $\xi$  and  $\eta$  to be two vectors in  $\mathcal{H}$ , subject to the restriction  $\|\xi\| \leq 1$ ,  $\|\eta\| \leq 1$ , but otherwise arbitrary. Now

$$|(\mathcal{F}^z(s)\xi, \eta)| \leq \|\mathcal{F}^z(s)\|_{\infty} \leq \|\mathcal{F}^z(s)\|_q.$$

Hence if we let

$$\Phi(s) = 1/A(\mathcal{F}^z(s)\xi, \eta),$$

then

$$(8.17) \quad \int_{-\infty}^{\infty} |\Phi(\sigma_1 + it)|^q dt \leq 1, \quad \int_{-\infty}^{\infty} |\Phi(1 - \sigma_1 + it)|^q dt \leq 1.$$

However  $\Phi(s)$  is clearly analytic in an open region containing  $\sigma_1 \leq R(s) \leq 1 - \sigma_1$ —it is analytic in  $0 < R(s) < 1$ . Moreover, it satisfies the growth condition specified in Lemma 11, with  $c = \frac{1}{2}$ . We also notice  $\sigma_1 < \frac{1}{2} < 1 - \sigma_1$ .

We then conclude:

$$\sup_{-\infty < t < \infty} |\Phi(\tfrac{1}{2} + it)| \leq A_{\sigma_1, q}.$$

Going back to our definition this leads to

$$\sup_{-\infty < t < \infty} |(\mathcal{F}^{\pm}(\tfrac{1}{2} + it)\xi, \eta)| \leq B_q.$$

Notice that  $A_{\sigma_1, q}$  and hence  $B_q$  is independent of  $\xi, \eta$ . Taking the sup over all  $\xi, \eta$ ,  $\|\xi\| \leq 1$ ,  $\|\eta\| \leq 1$  we obtain

$$\sup_{-\infty < t < \infty} \|\mathcal{F}^{\pm}(\tfrac{1}{2} + it)\| \leq B_q.$$

If we now drop the normalization  $\|f\|_p = 1$ , we obtain the conclusion of Corollary 2. This concludes the proof.

*Proof of Corollary 3.* The proof is similar to that of Corollary 2 but is somewhat more complicated.

Let  $f$  be a simple function. We use inequality (8.2) which we have already proved for such  $f$ . We fix some  $\delta > 0$ , and assume momentarily that  $\|f\|_p = 1$ . Let us call

$$\lambda = 1 - |\sigma_1 - \tfrac{1}{2}|q - \delta,$$

where we choose  $1/q < \sigma_1 < \frac{1}{2}$ .

Then (8.2) becomes

$$\int_{-\infty}^{\infty} \|\mathcal{F}^{\pm}(\sigma_1 + it)\|_q^q (1 + |t|)^{\lambda} dt \leq (A_{q, \sigma_1, \lambda})^q.$$

Choose  $\xi, \eta \in \mathcal{H}$ , with  $\|\xi\| \leq 1$ ,  $\|\eta\| \leq 1$ , and let

$$\Psi(s) = (\mathcal{F}^{\pm}(s), \xi, \eta).$$

Then

$$|\Psi(\sigma + it)| \leq \|(\mathcal{F}^{\pm}(\sigma + it)\xi, \eta)\| \leq \|\mathcal{F}^{\pm}(\sigma + it)\|_q.$$

The above then becomes

$$\int_{-\infty}^{\infty} |\Psi(\sigma_1 + it)|^q (1 + |t|)^{\lambda} dt \leq (A_{q, \sigma_1, \lambda})^q.$$

Since the formula (8.2) is symmetric in  $\sigma_1$  and  $1 - \sigma_1$ , one also obtains

$$\int_{-\infty}^{\infty} |\Psi(1 - \sigma_1 + it)|^q (1 + |t|)^\lambda dt \leq (A_{q, \sigma_1, \lambda})^q.$$

We let

$$\Phi(s) = c_1 (2 + s)^{\lambda/q} \Psi(s).$$

If we choose  $c_1$  as appropriate constant (depending on  $q$ ,  $\sigma_1$ , and  $\lambda$ ) then the above inequalities become

$$\int_{-\infty}^{\infty} |\Phi(\sigma_1 + it)|^q dt \leq 1, \text{ and } \int_{-\infty}^{\infty} |\Phi(1 - \sigma_1 + it)|^q dt \leq 1.$$

Moreover it is an easy matter to verify that  $\Phi(s)$  satisfies the growth condition specified in Lemma 11. We may thus conclude, (see (8.15)),

$$|\Phi(\sigma + it)| \leq c_2 [|\sigma - \sigma_1|^{-1/q} + |1 + \sigma - \sigma_1|^{-1/q}], \\ \sigma_1 < \sigma < 1 - \sigma_1.$$

Going back to the definitions of  $\Phi$  and  $\Psi$  the above becomes

$$|2 + s|^{\lambda/q} |\mathcal{F}^z(s, \xi, \eta)| \leq c_3 [|\sigma - \sigma_1|^{-1/q} + |1 - \sigma_1 - \sigma|^{-1/q}], \\ s = \sigma + it, \quad \sigma_1 < \sigma < 1 - \sigma_1.$$

Notice that the right-hand side is independent of  $\xi$ , and  $\eta$ . If we remember that  $\xi$  and  $\eta$  are arbitrary except  $\|\xi\| \leq 1$ ,  $\|\eta\| \leq 1$ , and we take the sup of the left-hand side, dropping the restriction  $\|f\|_p = 1$ , we then obtain

$$(8.18) \quad |2 + s|^{\lambda/q} \|\mathcal{F}^z(s)\| \leq c_3 [|\sigma - \sigma_1|^{-1/q} + |1 - \sigma_1 - \sigma|^{-1/q}] \|f\|_p,$$

where  $s = \sigma + it$ ,  $\sigma_1 < \sigma < 1 - \sigma_1$ ,  $1/q < \sigma_1 < \frac{1}{2}$ ,

$$c_3 = c_3(q, \sigma_1, \lambda).$$

Notice that this formula actually holds for every  $s$  in the open strip  $1/q < R(s) < 1/p$ . In fact, for such an  $s$ , we need only choose a  $\sigma_1$  so that

$$\sigma_1 < \sigma < 1 - \sigma_1, \text{ and } 1/q < \sigma_1 < \frac{1}{2}.$$

If we now fix our  $\lambda$  and  $s$ , it is clear that (8.18) implies Corollary 3.

*Proof of Corollary 4.* It is clear from Corollary 3, that whenever  $1/q < R(s) < 1/p$ ,  $\mathcal{F}^z(s)$  has a unique bounded extension to all of  $L_p(G)$ .

Inequality (8.18) shows that the bounds are uniform whenever  $s$  is restricted to a compact subset of  $1/q < R(s) < 1/p$ . But we know that  $\mathcal{F}^z$  is analytic in the strip  $0 < R(s) < 1$ , when  $f \in L_1(G) \cap L_p(G)$ .

Hence a simple limiting argument also shows that  $\mathcal{F}^\pm$  is analytic in  $1/q < R(s) < 1/p$ , for each fixed  $f \in L_p(G)$ .

Other limiting arguments (which we will not give) show that the extension  $\mathcal{F}^\pm$  to all of  $L_p$  also satisfies the inequalities (8.2) and those contained in Corollaries 1, 2, and 3.

This concludes our discussion of Corollary 4.

**9. The discrete series.** We now intend to investigate the form of the Hausdorff-Young theorem for our group, so far as it involves the discrete series.

As contrasted with the case of the continuous series considered above, we do not concern ourselves with an analytic structure in the discrete series. This lack is mitigated by the fact that in the Plancherel formula for the group, elements of the discrete series occur with weights bounded away from zero.

We begin by proving the following theorem.

**THEOREM 8.** *Let  $1 \leq p \leq 2$ , and  $1/q + 1/p = 1$ . Then*

$$(9.1) \quad \left( \sum_{k=0}^{\infty} (k + \frac{1}{2}) \|D^+(f, k)\|_{q^q}^q + (k+1) \|D^-(f, k)\|_{q^q}^q \right)^{1/q} \leq \|f\|_p$$

*whenever  $f \in L_1(G) \cap L_p(G)$ .*

*Proof.* We consider the measure space  $M$ , defined as follows: The points of  $M$  are the pairs  $(k, \pm)$ , where  $k$  runs over the non-negative integers, and the second component is either  $+$  or  $-$  as indicated.

On  $M$  we define the measure  $dm$  as follows: The point  $(k, +)$  is assigned the measure  $k + \frac{1}{2}$ ; the point  $(k, -)$  is assigned the measure  $k + 1$ .

We let  $\mathcal{H}$  denote a separable infinite-dimensional Hilbert space. In accordance with the discussion of § 2 we consider functions from  $M$  to bounded operators on  $\mathcal{H}$ . In view of the discreteness of  $M$ , all such functions are automatically measurable.

We now define a mapping from simple functions on  $G$  to operator valued functions on  $M$ . The mapping, which we denote by  $T$ , is given by

$$T: f \rightarrow F = \{F(k, \pm)\},$$

$$F(k, \pm) = D^\pm(f, k),$$

and with

$$D^\pm(f, k) = \int_G D^\pm(g, k) f(g) dg.$$

As explicitly given, the representations  $D^\pm(\cdot, k)$  act on different Hilbert

spaces. However, since all separable infinite dimensional Hilbert spaces are unitarily equivalent, we may assume that we deal with appropriate unitarily equivalent representations, all of which act on our given  $\mathcal{H}$ .

Using the definitions of § 2, (9.1) becomes

$$(9.2) \quad \|T(f)\|_p \leq \|f\|_q.$$

This is what we must prove

Observe that by definition,

$$\|T(f)\|_2^2 = \sum_{k=0}^{\infty} (k + \frac{1}{2}) \|D^+(f, k)\|_2^2 + (k + 1) \|D^-(f, k)\|_2^2.$$

Hence, in view of the Plancherel formula for  $G$ , (see § 5), we have

$$(9.3) \quad \|T(f)\|_2 \leq \|f\|_2.$$

Notice also that

$$\|T(f)\|_{\infty} = \sup_{k, \pm} \|D^{\pm}(f, k)\|_{\infty}$$

while

$$\|D^{\pm}(f, k)\|_{\infty} \leq \|f\|_1,$$

since  $D^{\pm}(\cdot, k)$  is unitary. We therefore have,

$$(9.4) \quad \|T(f)\|_{\infty} \leq \|f\|_1.$$

We now use the general interpolation theorem of § 3. In the present case the operator  $T$  is independent of  $z$ , and so *a fortiori* satisfies the conditions of analyticity and admissible growth.

We let  $(p_0, q_0) = (2, 2)$ , and  $(p_1, q_1) = (1, \infty)$ . Then it is easily verified that  $1/p + 1/q = 1$ , and that we may choose any  $p$ ,  $1 \leq p \leq 2$ , by an appropriate choice of  $\tau$ ,  $0 \leq \tau \leq 1$ .

It is apparent that in the present case  $A_0(y) = 1$  because of (9.3), and also  $A_1(y) = 1$  because of (9.4).

The result of Theorem 3 is

$$\|T(f)\|_q \leq A_{\tau} \|f\|_p.$$

Since  $A_0(y) = A_1(y) = 1$ , it follows that  $\log A_{\tau} = 0$ , and hence  $A_{\tau} = 1$ . Thus we have demonstrated (9.2), and therefore (9.1), whenever  $f$  is a simple function.

The extension of the inequality to all  $L_1(G) \cap L_p(G)$  follows by standard limiting arguments. This concludes the proof of the theorem.

The following corollary is basic for our applications of the above theorem.

**COROLLARY.** *The mapping  $f \rightarrow D^z(f, k)$  has a unique extension to all of  $L_p(G)$ , and this extension satisfies the following:*

$$(9.5) \quad \sup_{k, z} \|D^z(f, k)\|_\infty \leq 2^{1-1/p} \|f\|_p$$

whenever  $1 \leq p \leq 2$ .

*Proof.* We consider first the case when  $f \in L_1 \cap L_p$ . Using (9.1) we obtain

$$(k + \tfrac{1}{2}) \|D^+(f, k)\|_{q^q} \leq \sum_{k=0}^{\infty} (k + \tfrac{1}{2}) \|D^+(f, k)\|_{q^q} \leq \|f\|_{p^q}.$$

Hence,

$$\|D^+(f, k)\|_{q^q} \leq 1/(k + \tfrac{1}{2}) \|f\|_p \leq 2 \|f\|_{p^q}.$$

A similar argument for  $D^-(f, k)$  shows that

$$\sup_{k, z} \|D^z(f, k)\|_q \leq 2^{1/q} \|f\|_p = 2^{1-1/p} \|f\|_p.$$

Since the operator norms used above are non-increasing with  $q$ , (see (2.2)), we conclude (9.5), whenever  $f \in L_1 \cap L_p$ .

In view of the inequality just proved it follows that the mapping  $f \rightarrow D^z(f, k)$  has a unique extension to  $L_p$  which again satisfies (9.5).

#### CHAPTER IV. APPLICATIONS.

**10. Boundedness of convolution operator.** We are now in a position to obtain an important application of the analysis of the previous sections.

We shall find it convenient to adopt a slight change in our notation. In this section letters  $x, y, z, \dots$  will denote elements of the group  $G$ , and  $f, g, h, \dots$  functions on the group.

We recall the operation of convolution of two functions  $f$  and  $g$ , defined as follows

$$(f * g)(x) = \int_G f(xy^{-1})g(y)dy,$$

$$dy = \text{Haar measure.}$$

Now if  $f \in L_2$ , and  $g \in L_p$ ,  $1 \leq p \leq 2$ , then by Young's inequality (see [23]),  $f * g$  is well defined and is in  $L_r$ , where  $1/r = \frac{1}{2} + 1/p - 1$ .

**THEOREM 9.** *Let  $f \in L_2$ , and  $g \in L_p$ ,  $1 \leq p < 2$ ; if  $h = f * g$ , then  $h \in L_2$ , and*

$$(10.1) \quad \|h\|_2 \leq A_p \|f\|_2 \|g\|_p,$$

where  $A_p$  does not depend on  $f$  or  $g$ . Hence the operation of convolution by a function  $g \in L_p$ ,  $1 \leq p < 2$ , is a bounded operator on  $L_2$ .

*Remarks.* Inequality (10.1) fails when  $p = 2$ . This is not surprising for many reasons; we indicate one such reason. Inequality (10.1) is essentially a statement of the fact that the Fourier transform of a function  $g$  in  $L_p$ ,  $1 \leq p < 2$ , is uniformly bounded. But a function in  $L_2$  may be given by appropriately assigning its Fourier transform, and this may be done so that the Fourier transform is not uniformly bounded.

The statement which corresponds to (10.1) when  $G$  is, for example, a non-compact abelian group is false, as long as  $p \neq 1$ . This is so even in the case when  $G$  is the additive group of the real-line. We postpone further discussions of these matters to the next section.

*Proof.* It is sufficient to prove inequality (10.1) for a dense class of functions, and so we assume that  $f$  and  $g$  are in  $L_1$  (in addition to the fact that  $f$  and  $g$  are respectively also in  $L_2$  and  $L_p$ ).

Notice that if  $h = f * g$ , and  $x \rightarrow U_x$  is any (say unitary) representation, then

$$(10.2) \quad U_h = U_f \cdot U_g.$$

Here  $U_f = \int_G f(x) U_x dx$ , with similar definitions for  $U_g$ , and  $U_h$ .

Moreover by (2.13) and (10.2) we obtain

$$(10.3) \quad \|U_h\|_2 \leq \|U_f\|_2 \|U_g\|_\infty.$$

We apply (10.3) successively to the cases when  $U = U^+(\cdot, \frac{1}{2} + it)$ , (the continuous principal series), and  $U = D^+(\cdot, k)$ , (the discrete series).

For the continuous principal series we apply Corollary 2 of Theorem 7, (with  $g$  in place of  $f$ ); for the discrete series we similarly apply the corollary of Theorem 8. The result for the continuous series is

$$(10.4) \quad \|U^+(h, \tfrac{1}{2} + it)\|_2 \leq A_p \|U^+(f, \tfrac{1}{2} + it)\|_2 \|g\|_p, \\ 1 \leq p < 2, \text{ with } A_p \text{ independent of } t.$$

The result for the discrete series is

$$(10.5) \quad \|D^+(h, k)\|_2 \leq 2^{1-1/p} \|D^+(f, k)\|_2 \|g\|_p, \quad 1 \leq p \leq 2.$$

Finally, we calculate  $\|h\|_2$  and  $\|f\|_2$  via the Plancherel formula, (see § 5).

It is to be noted that in computing the required norms, it makes no difference whether we use the representations  $v^*(\cdot, \frac{1}{2} + it)$ , or the unitarily equivalent representations  $U^*(\cdot, \frac{1}{2} + it)$ . Using (10.4) and (10.5) we then easily obtain

$$\|h\|_2 \leq A_p \|f\|_2 \|g\|_p, \quad 1 \leq p < 2.$$

This proves (10.1), and hence the theorem.

From the above theorem, and with the use of various devices, it is possible to prove other inequalities like (10.1). All of these have in common the remarkable property that they hold for the group we are considering and also for compact groups, but fail in the simplest non-compact abelian instances. We shall limit ourselves to the proof of only one more such result.

**COROLLARY.** Let  $f \in L_2$ , and  $g \in L_2$ . If  $h = f * g$ , then  $h \in L_q$ ,  $2 < q \leq \infty$ , and

$$(10.6) \quad \|h\|_q \leq A_q \|f\|_2 \|g\|_2,$$

where  $A_q$  does not depend on  $f$  or  $g$ .

*Remark.* By the results of the next section it will be seen that this corollary and the theorem from which it is derived are essentially equivalent results.

*Proof.* Let  $k \in L_1 \cap L_p$ , where  $1/p + 1/q = 1$ , but let  $k$  be arbitrary otherwise. Then,

$$\begin{aligned} \int_G h(x) k(x) dx &= \int_G k(x) \int_G f(xy^{-1}) g(y) dy dx \\ &= \int_G g(y) \int_G k(x) f(xy^{-1}) dx dy = \int_G g(y) l(y) dy, \end{aligned}$$

where  $l = \tilde{f} * k$ , with  $f^*(x) = \tilde{f}(x^{-1})$ .

Hence,

$$\left| \int_G h(x) k(x) dx \right| = \left| \int_G g(y) l(y) dy \right| \leq \|g\|_2 \|l\|_2.$$

However, by our theorem

$$\|l\|_2 \leq A_p \|f\|_2 \|k\|_p,$$

since  $1 \leq p < 2$ . Thus we have

$$\left| \int_G h(x) k(x) dx \right| \leq A_p \|f\|_2 \|g\|_2 \|k\|_p.$$

Now take the sup of the left-hand side over  $k$ , such that  $\|k\|_p = 1$ . The result is

$$\|h\|_q \leq A_p \|f\|_2 \|g\|_2,$$

and the corollary is proved.

**11. Characterization of unitary representations of  $G$ .** Let  $g \rightarrow U_g$  be a unitary representation (not necessarily irreducible) on a Hilbert space  $\mathcal{H}$ . We now introduce two notions which are basic for our characterization of the representations of  $G$ .

*Definition.*  $\phi(g)$  is an *entry function*, if

$$(11.1) \quad \phi(g) = (U_g \xi, \eta), \quad \xi, \eta \in \mathcal{H}.$$

*Definition.*  $g \rightarrow U_g$  is *extendable to  $L_p(G)$* , if for some fixed  $p$ ,  $p \geq 1$ ,

$$(11.2) \quad \|U_f\|_\infty \leq A \|f\|_p, \text{ every } f \in L_1(G) \cap L_p(G),$$

with  $A$  independent of  $f$ .

It is interesting to note the following facts. Theorem 9, which dealt with the boundedness of the operation of convolution, can be restated by saying that the *regular* representation is extendable to  $L_p(G)$  for every  $p$ ,  $1 \leq p < 2$ . We may further note that the corollary to Theorem 9 states that every entry function of the regular representation is in  $L_q(G)$ , for every  $q > 2$ .

As a preliminary matter, we obtain the following relation between the notions defined above.

**LEMMA 27.<sup>18</sup>** *The representation  $g \rightarrow U_g$  is extendable to  $L_p(G)$ , if and only if for every pair  $\xi, \eta \in \mathcal{H}$ , the entry function  $\phi$ , defined in (11.1), lies in  $L_q(G)$ , where  $1/p + 1/q = 1$ .*

Assume first that  $U_g$  is extendable to  $L_p$ . Let  $f \in L_1 \cap L_p$ . Then

$$\begin{aligned} \int_G \phi(g) f(g) dg &= \int (U_g \xi, \eta) f(g) dy \\ &= \left( \int U_g f(g) dg \xi, \eta \right) = (U_f \xi, \eta). \end{aligned}$$

Thus by (11.2)

$$\left| \int_G \phi(g) f(g) dg \right| \leq |(U_f \xi, \eta)| \leq A_p \|f\|_p \|\xi\| \|\eta\|.$$

<sup>18</sup> This lemma holds for any locally compact group.

We now limit ourselves to those  $f$ 's for which  $\|f\|_p = 1$ , and we take the sup of the left-hand side. We then obtain  $\|\phi\|_q \leq A_p \|\xi\| \|\eta\|$ , and thus  $\phi \in L_q$ . This proves the implication in one direction. To prove the converse we shall use the closed-graph theorem several times. We argue as follows.

For fixed  $\eta$ , consider the mapping

$$\xi \rightarrow (U_\sigma \xi, \eta) = \phi(g)$$

as a mapping from  $\mathcal{A}$  to  $L_q(G)$ . By the assumptions of the lemma, it is clear that this mapping is everywhere defined on  $\mathcal{A}$ ; obviously it is linear. We next notice that it is closed. For suppose that  $\xi_n \rightarrow \xi$ , and

$$\phi_n(g) = (U_\sigma \xi_n, \eta) \rightarrow \phi_0(g) \text{ in } L_q \text{ norm.}$$

However,  $\phi_n(g) \rightarrow \phi(g) = (U_\sigma \xi, \eta)$ , for every  $g \in G$ . Thus  $\phi(g) = \phi_0(g)$  a. e., and  $\phi_n(g) \rightarrow \phi(g)$  in  $L_q$  norm. This shows that the mapping is closed.

Hence,

$$(11.3) \quad \|\phi\|_q \leq A_\eta \|\xi\|.$$

Similarly,

$$(11.4) \quad \|\phi\|_q \leq B_\xi \|\eta\|.$$

Now let  $f$  be any function in  $L_p(G)$ . We propose to define  $U_f$ . We shall do this by defining  $(U_f \xi, \eta)$ , for every pair  $\xi, \eta \in \mathcal{A}$ .

In fact set

$$(U_f \xi, \eta) = \int_G \phi(g) f(g) dg,$$

where  $\phi(g) = (U_\sigma \xi, \eta)$ . Since  $\phi \in L_q$ ,  $f \in L_p$ , and  $1/p + 1/q = 1$ , the integral is well-defined, by Hölder's inequality. Hölder's inequality, (11.3), and (11.4) further show:

$$(11.5) \quad |(U_f \xi, \eta)| \leq A_\eta \|\xi\| \|f\|_p,$$

and

$$(11.6) \quad |(U_f \xi, \eta)| \leq B_\xi \|\eta\| \|f\|_p.$$

Now (11.6) shows that the vector  $U_f \xi$  is well-defined for every  $\xi \in \mathcal{A}$ . Moreover, (11.5) and a simple argument, prove that  $U_f$  is a closed operator. Hence, using the closed graph theorem, we obtain that  $U_f$ , for each  $f \in L_p(G)$ , is a bounded operator on  $\mathcal{A}$  (to itself).

Finally, consider the mapping

$$f \rightarrow U_f,$$

which is a mapping from  $L_p(G)$  to  $\mathcal{B}$  = Banach space of bounded operators

on  $\mathcal{H}$  with usual norm. We have just seen that this mapping is everywhere defined. It is clear from the definition that this mapping is linear. We shall next see that it is closed. In fact, assume  $f_n \rightarrow f$  in  $L_p$  norm, and that  $U_{f_n} \rightarrow U_0$  in the operator norm. Then

$$(U_{f_n}\xi, \eta) \rightarrow (U_0\xi, \eta)$$

for every  $\xi$  and  $\eta \in \mathcal{H}$ . By (11.5) it follows that

$$(U_{f_n}\xi, \eta) \rightarrow (U_f\xi, \eta).$$

Hence

$$(U_f\xi, \eta) = (U_0\xi, \eta).$$

Thus  $U_f = U_0$ . Therefore the mapping  $f \rightarrow U_f$  is closed. A final application of the closed graph theorem gives

$$\|U_f\| \leq A \|f\|_p.$$

This shows that  $g \rightarrow U_g$  is extendable to  $L_p$ , and the lemma is completely proved.

We notice that the lemma proves that Theorem 9 and its corollary are equivalent propositions. It is to be observed that the identity representation (on the one-dimensional space) is *not* extendable to  $L_p$  if  $p \neq 1$ . Thus there are very simple representations which are not extendable to  $L_p$  if  $p \neq 1$ . We make one further remark before we proceed. Every entry function is automatically in  $L_\infty(G)$ . Hence a simple argument shows that if it is in  $L_{q_0}(G)$ , it is also in  $L_q(G)$ , where  $q > q_0$ . Therefore the lemma leads to the fact that if a representation is extendable to  $L_{p_0}(G)$ , and  $p_0 > 1$ , then it also is extendable to  $L_p(G)$ , for  $1 \leq p < p_0$ .

We are now in a position to give our characterization of the irreducible unitary representations of the  $2 \times 2$  real unimodular group  $G$ .

**THEOREM 10.** *Let  $g \rightarrow U_g$  be an irreducible unitary representation of  $G$ . Assume  $U$  is not the identity representation. Then*

(a)  *$U$  is unitarily equivalent to an element of the discrete series if and only if  $U$  is extendable to  $L_2(G)$ .*

(b)  *$U$  is unitarily equivalent to an element of the continuous principal series if and only if  $U$  is extendable to every  $L_p(G)$ ,  $1 \leq p < 2$ , but is not extendable to  $L_2(G)$ .*

(c)  *$U$  is unitarily equivalent to the element of the complementary series corresponding to the parameter  $\sigma$ ,  $0 < \sigma < \frac{1}{2}$ , if and only if  $U$  is*

extendable to  $L_p(G)$  for  $1 \leq p < 1/(1-\sigma)$ , but is not extendable to  $L_{1/(1-\sigma)}(G)$ .

**COROLLARY.** *Let  $g \rightarrow U_g$  be an irreducible unitary representation different from the identity representation. Then  $U$  is unitarily equivalent to (1) an element of the discrete series, (2) an element of the continuous principal series, or, (3) the element of the complementary series corresponding to  $\sigma$ ,  $0 < \sigma < \frac{1}{2}$ , if and only if respectively (1') every entry function is in  $L_2(G)$ , (2') every entry function is in  $L_q(G)$ ,  $q > 2$ , but not every entry function is in  $L_2(G)$ , or, (3') every entry function is in  $L_q(G)$ ,  $q > 1/\sigma$ , but not every entry function is in  $L_{1/\sigma}(G)$ .*

Before we pass to the proof of these facts we should like to clarify the difference of notation that we have adopted for the representations of  $G$  and that which is used in Bargmann's paper. The parameter  $\sigma$ ,  $0 < \sigma < \frac{1}{2}$ , which we have used to identify the elements of the complementary series corresponds to Bargmann's parameter  $\frac{1}{2} - \sigma$ . There is also a difference in parametrization of the discrete series. We have called elements of the discrete series those which appear as discrete summands (with non-zero measure) in the Plancherel formula of the group. This exhausts Bargmann's discrete series, except for the representations which he labels  $D_{1/2}^+$  and  $D_{1/2}^-$ . In our notation these two elements occur as follows. The representation  $g \rightarrow U^-(g, \frac{1}{2})$  of the continuous principal series is not irreducible. It splits into the direct sum of  $D_{1/2}^+$  and  $D_{1/2}^-$ . Thus in our notation we count  $D_{1/2}^+$  and  $D_{1/2}^-$  as elements of the continuous principal series. It is with these definitions in mind that the above theorem and corollary are stated.

Now to the proof. It is known that every irreducible unitary representation of the group is, except for the trivial representation, up to the unitary equivalence, either an element of the discrete series, the continuous principal series, or the complementary series.

By the corollary of Theorem 8, it follows that elements of the discrete series are extendable to  $L_2(G)$ . By Corollary 2 of Theorem 7 it follows that elements of the continuous principal series are extendable to  $L_p(G)$ ,  $1 \leq p < 2$ . If we consider the representation  $g \rightarrow U^-(g, \frac{1}{2})$  we see that it is also extendable to  $L_p$ , for  $1 \leq p < 2$ . However, this representation splits into two irreducible representations (which we have counted among the continuous principal series). A simple argument shows that each of these pieces is then also extendable to  $L_p(G)$ ,  $1 \leq p < 2$ .

Finally, Corollary 3 of Theorem 7 implies that the element corresponding to  $\sigma$ ,  $0 < \sigma < \frac{1}{2}$ , is extendable to  $L_p$ ,  $1 \leq p < 1/(1-\sigma)$ . We must now show

that elements of the continuous principal series are not extendable to  $L_2(G)$ , and the element of the complementary series corresponding to  $\sigma$  is not extendable to  $L_{1/(1-\sigma)}(G)$ . We consider first the continuous principal series. By Lemma 27 it is sufficient to exhibit an entry function which is not in  $L_2(G)$ . We consider the parametrization of the group given by Bargmann with  $0 < y < \infty$ ,  $0 \leq \mu \leq 2\pi$ , and  $0 \leq \nu < 2\pi$ . In this case Haar measure becomes  $(2\pi)^{-2} dy d\mu d\nu$ , (see Bargmann (10.14)). We consider the "principal spherical function" corresponding to this representation. In Bargmann's notation this is  $W_{00}(y)$ , which has the asymptotic expansion, as  $y \rightarrow \infty$ ,

$$W_{00}(y) \sim 2y^{-\frac{1}{2}} R(\beta_{00}(it, 0)y^{it}).$$

We also have

$$|\beta_{0,0}(it, 0)|^2 = (\coth \pi t)/4\pi t,$$

or  $(\tanh \pi t)/4\pi t$ , depending on whether we are dealing with  $U^+(g, \frac{1}{2} + it)$  or  $U^-(g, \frac{1}{2} + it)$ , (see Bargmann (11.4), (11.7<sup>a</sup>), and (11.7<sup>b</sup>)). These asymptotic relations are valid except for  $U^-(g, \frac{1}{2})$ . Except for this case we can easily see that the element  $W_{00}(y)$  is not in  $L_2(0, \infty; dy)$ , because of the factor  $y^{-\frac{1}{2}}$ . Thus the corresponding principal spherical functions are not in  $L_2(G)$ .

In considering the representation  $U^-(g, \frac{1}{2})$  we recall that it splits into  $D^+_{\frac{1}{2}}$  and  $D^-_{\frac{1}{2}}$  (in Bargmann's notation). It is also demonstrated by Bargmann that the spherical functions corresponding to these representations are asymptotic to constant times  $y^{-\frac{1}{2}}$ . Thus these are also not in  $L_2(G)$ . The complementary series is dealt with similarly. Taking into account our difference in notation, we have  $\beta_{00}(\frac{1}{2} - \sigma, 0)y^{-\sigma}$ , as asymptotic expression (as  $y \rightarrow \infty$ ) for the principal spherical function corresponding to  $\sigma$ , (see Bargmann (11.5)). Now clearly this function is not in  $L_{1/\sigma}(0, \infty; dy)$ . Hence the principal spherical function is not in  $L_{1/\sigma}(G)$ , and thus the representation is not extendable to  $L_{1/(1-\sigma)}(G)$ . If we recall Lemma 27, we see that Theorem 10 and its corollary are completely demonstrated.

We now pass to the consideration of not necessarily irreducible unitary representations. Let  $g \rightarrow U_g$  be such a representation of our group on a separable Hilbert space  $\mathcal{H}$ .

Using the von Neuman reduction theory [17], and following Segal [19], we may decompose the representation as follows.

The Hilbert space  $\mathcal{H}$  may be written as a direct integral  $\int_{\oplus} \mathcal{H}^{\lambda} d\sigma(\lambda)$  of Hilbert spaces  $\mathcal{H}^{\lambda}$ . With respect to this decomposition, the representation  $g \rightarrow U_g$  may be decomposed into  $\{U^{\lambda}_g\}$ , where  $g \rightarrow U^{\lambda}_g$  is irreducible and unitary, for a.e.  $\lambda$ .

We do not wish to go into the background of these facts, or into the sense in which this reduction is unique. Aside from the simple manipulative facts which we shall use, we shall also use the following fact: Let  $A$  be an operator on  $\mathcal{H}$  which can be decomposed with respect to the above decomposition of  $\mathcal{H}$  into the direct integral of the  $\mathcal{H}^\lambda$ 's. We write  $A = \{A^\lambda\}$ . Then  $\|A\|_\infty = \operatorname{ess\,sup}_\lambda \|A^\lambda\|_\infty$ .

Our theorem is the following: It may be viewed as an extension and clarification of Theorem 9 and its corollary.

**THEOREM 11.** *Let  $g \rightarrow U_g$  be a unitary representation of  $G$  on  $\mathcal{H}$ . Consider its reduction into a direct integral of irreducible unitary representations  $g \rightarrow U^\lambda_g$ . A necessary and sufficient condition that (except for a set of measure zero) every  $U^\lambda_g$  be unitarily equivalent to elements of the discrete or continuous principal series is that the representation  $g \rightarrow U_g$  be extendable to  $L_p(G)$  for every  $p$ ,  $1 \leq p < 2$ . Alternatively, the condition is equivalent with requiring that every entry function of the representation  $g \rightarrow U_g$  be in every  $L_q(G)$ ,  $2 < q$ .*

*Proof.* Assume first that, disregarding a set of measure zero, every  $U^\lambda_g$  is equivalent to either elements of the discrete series or of the continuous principal series. Let  $f \in L_1(G) \cap L_p(G)$ . Then

$$U_f = \{U^{\lambda_f}\}.$$

Now  $\|U_f\|_\infty = \operatorname{ess\,sup}_\lambda \|U^{\lambda_f}\|_\infty$ . Because of Corollary 2 of Theorem 7, and the corollary to Theorem 8, we obtain

$$\operatorname{ess\,sup}_\lambda \|U^{\lambda_f}\|_\infty \leq A_p \|f\|_p, \quad 1 \leq p < 2;$$

we have disregarded the set of measure zero which does not correspond to either the discrete or continuous principal series. Hence,

$$\|U_f\| \leq A_p \|f\|_p, \quad 1 \leq p < 2,$$

and  $g \rightarrow U_g$  is extendable to every  $L_p(G)$ ,  $1 \leq p < 2$ . To prove the converse, we argue as follows. Let  $\{f_n\}$  be a denumerable collection of functions on  $G$  which lie in every  $L_p(G)$ , and are dense in every  $L_p(G)$ ,  $1 \leq p < \infty$ . Now  $U_{f_n} = U^{\lambda_{f_n}}$ . Since  $g \rightarrow U_g$  can be extended to  $L_p(G)$ , we have

$$\|U_{f_n}\|_\infty \leq A_p \|f_n\|_p, \quad 1 \leq p < 2.$$

Thus,

$$\operatorname{ess\,sup}_\lambda \|U^{\lambda_{f_n}}\|_\infty \leq A_p \|f_n\|_p, \quad 1 \leq p < 2.$$

Let  $E_n$  be the exceptional set of measure zero corresponding to the above

Let  $E = \bigcup E_n$ ; then  $E$  is still of measure zero. Now

$$\sup_{\lambda \notin E} \|U^{\lambda}_{f_n}\|_{\infty} \leq A_p \|f_n\|_p, \quad 1 \leq p < 2.$$

Owing to the denseness of the collection  $\{f_n\}$ , we obtain

$$\|U^{\lambda}_f\|_{\infty} \leq A_p \|f\|_p, \quad 1 \leq p < 2, \lambda \notin E, f \in L_1 \cap L_p.$$

Therefore  $U^{\lambda}_g$  can be extended to  $L_p$ ,  $1 \leq p < 2$ , for every  $\lambda \notin E$ . By Theorem 10, and the fact that  $E$  is a set of measure zero we obtain that almost every  $U^{\lambda}_g$  belongs to either the discrete or continuous principal series. Using Lemma 27 we obtain the alternate condition.

This concludes the proof of Theorem 11.

Let us now consider the additive group of the line. We shall show that the analogues of Theorem 9 fails for every  $p \neq 1$ , and that the analogue of the corollary of Theorem 9 fails if  $q \neq \infty$ .

In fact let  $f(x) = (\log(2 + |x|))^{-1}$ ,  $-\infty < x < \infty$ . Then by the use of Theorem 124 of Titchmarsh [21] it may be shown  $f$  is the Fourier transform of a positive function which is in  $L_1(-\infty, \infty)$ . A simple application of the Plancherel theorem then shows that  $f$  is the convolution of two functions in  $L_2(-\infty, \infty)$ . However, clearly,  $f \notin L_q(-\infty, \infty)$ , if  $q \neq \infty$ . Thus the analogue of the corollary of Theorem 9 fails. Because of Lemma 27 applied to the regular representation on  $L_2(-\infty, \infty)$ , we see that the analogue of Theorem 9 fails if  $p \neq 1$ .

Let us consider the problem of whether Theorem 9 would hold for our group in the case  $p = 2$ . This, clearly, is equivalent to requiring that the regular representation of the group is extendable to  $L_2(G)$ . By an argument like that in the proof of Theorem 11, it would then follow that the regular representation can be written as a direct sum of representations equivalent to representations of the discrete series. This, of course, is not true.

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# ON MAXIMAL FIRST ORDER PARTIAL DIFFERENTIAL OPERATORS.\*

By H. O. CORDES.<sup>1</sup>

The modern spectral theory of selfadjoint differential operators, in all its various presentations, usually has the following starting point. One considers a linear formally selfadjoint differential operator  $L$  acting on either scalar or vector functions defined in some domain  $D$ . Then one defines a minimal and a maximal operator  $L_0$  in  $\mathfrak{D}(L_0)$  and  $L_1$  in  $\mathfrak{D}(L_1)$  to be the closure of the operator  $L$  in the domain of all sufficiently smooth functions vanishing in some neighborhood of the boundary and the (strict) adjoint of this operator respectively. Clearly  $L_0$  is hermitian symmetric and is a restriction of  $L_1$ :

$$(1) \quad L_0 \subset L_1.$$

Both operators  $L_0$  and  $L_1$  are closed operators of a certain appropriate  $L_2$ -space into itself. Now, by von Neumann's Theory about extensions of closed hermitian symmetric operators, there exist selfadjoint extensions of  $L_0$  (which also are restrictions of  $L_1$ ) if any additional condition imposed on the operator  $L$ , as for instance reality or semiboundedness, guarantees the v. Neumann defect indices to be equal. Then, any such extension  $M$  being given, it is proved that the spectral representation

$$(2) \quad M = \int \lambda dE_\lambda$$

of  $M$  leads to an integral representation of arbitrary functions  $f$  in terms of the (regular and singular) eigenfunctions of  $M$  which are solutions of the equation

$$(3) \quad Lu = \lambda u.$$

This way, for instance, has been used by Stone in his book [18] for second order ordinary differential operators; then by Kodaira [11], Levinson [12], Levitan [13], Coddington [4] and many others for  $n$ -th order ordinary

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operators; finally, for  $n$ -th order elliptic partial differential operators, by Garding [8], Vishik [19], [20], and Browder [1], and others. Now for ordinary differential operators a complete characterization of the possible operators  $M$  in  $\mathfrak{D}(M)$  is easily done because then the defect indices of  $L_0$  are finite and hence it easily can be shown that the domain  $\mathfrak{D}(M)$  can be characterized by imposing boundary conditions on the functions of  $\mathfrak{D}(L_1)$ . These boundary conditions are conditions in the usual sense if the boundaries are regular. On the other hand, for a partial differential operator in a domain with regular boundary, the operator  $L_0$  usually has infinite defect indices. This is why the characterization of the domain  $\mathfrak{D}(M)$  by boundary conditions becomes a problem which in general has not been solved up to now.

However, if we understand the word "boundary condition" in some weakened manner, a solution can be achieved and this has been done by several authors (Calkin [2], Vishik [19], [20], Hoermander [10], Phillips [14]). One then introduces a certain boundary space  $\mathfrak{B}$  which is the quotient space

$$(4) \quad \mathfrak{G}(L_1)/\mathfrak{G}(L_0),$$

where  $\mathfrak{G}(L_1)$  and  $\mathfrak{G}(L_0)$  denote the graphs of the operators  $L_1$  and  $L_0$  respectively. It is easy to see that the residue class of some element  $u \in \mathfrak{G}(L_1)$  with respect to the above quotient space depends only on the behavior of  $u$  at the boundary. Posing a generalized homogeneous boundary condition then simply means to impose on  $u$  that its residue class be in some given fixed subspace of  $\mathfrak{B}$ . One then can study the question whether or not a boundary condition is "well posed," i.e., leads to a selfadjoint (or more generally a maximal) operator  $M$ .

Nevertheless, it still remains the basic question to find out when such a generalized boundary condition can be posed in the usual way, i.e., imposing conditions on the function and its normal and tangential derivatives at the boundary only, and not in some neighborhood of the boundary. There are some more simple conditions, as for instance the Dirichlet condition, the von Neumann condition, and Hilbert's boundary condition of the third type, which have been investigated for certain types of differential operators. Also there are some nonlocal boundary conditions studied (Vishik [19], [20]), again for special types of differential operators.

The present paper deals with the discussion of a method which has many chances to work for very general classes of differential operators. We will discuss here maximality of formally selfadjoint first order partial differential operators acting on vector functions under boundary condition of a very large class which contains both local and nonlocal conditions. The conditions we

pose are boundary conditions in the strict sense mentioned above. It is interesting to see that ellipticity here does not play any further role.

In addition it seems to be quite obvious that similar arguments will work also in the case of higher order operators. The author is preparing another publication about the case of  $n$ -th order elliptic equations which will state quite analogous results.

The theory is based on investigating a relation between the inner product of the boundary space (which is essentially the inner product of the graph space) and another inner product which is defined for all "strips," i.e., all combinations of functions and sufficiently many normal derivatives at the boundary, under an ordinary  $L_2$ -norm with appropriate weight factor. The space completed under this norm will be called  $\mathfrak{E}$ . This relation will be established by a certain unbounded selfadjoint operator  $G$  in  $\mathfrak{D}(G)$  of the boundary space  $\mathfrak{B}$  into itself. The eigenvalue problem  $G\phi = \lambda\phi$  is very closely related to some kind of eigenvalue problems with the parameter in the boundary condition studied first by Hilbert [9] for the operator

$$(5) \quad L = (\partial/\partial x)p(\partial/\partial x) + (\partial/\partial y)p(\partial/\partial y)$$

and generalized to the case of a second order elliptic operator on a Riemannian space by Sandgren [17].

After having this operator  $G$  established it is possible to study also boundary conditions which are imposed in the space  $\mathfrak{E}$ , i.e., which are ordinary boundary conditions.

The abstract classifications of the possible boundary conditions in the generalized sense for the case of the operators studied here has been investigated in great detail by R. S. Phillips [14]. Phillips studies not only selfadjoint operators but also a class of operators he calls maximal dissipative. A maximal dissipative extension  $M$  of  $L_0$  simply is a maximal operator (in the v. Neumann sense) which essentially satisfies the condition

$$(6) \quad M + M^* \leq 0.$$

This class of operators is important because a maximal dissipative operator will be an infinitesimal generator of a semigroup of contractions. The selfadjoint case obviously arises if in (6) the equality sign holds. Then  $iM$  will be selfadjoint. Accordingly in this paper we shall look out for boundary conditions which will furnish maximal dissipative operators. Since the operator  $-L$  instead of  $L$  also can be considered, it then also will be possible to find conditions which make  $M$  and  $-M$  both maximal dissipative which amounts to  $iM$  being selfadjoint.

Finally it has to be mentioned that quite recently general locally dissipative conditions have been investigated by K. O. Friedrichs [7], P. D. Lax and R. S. Phillips [15]. This paper will essentially use the fact that a special operator  $M$  characterized by local boundary conditions is maximal dissipative. In other words we have to use the proofs mentioned above, which were achieved by use of mollifiers. The existence of the operator  $G$  mentioned above essentially follows from the well known result of K. O. Friedrichs [6] concerning identity of weak and strong solutions of first order systems and a certain continuation theorem proved in a concurrent paper of the author [5].

The main result is contained in Theorem 6.6. Section 2 contains the study of the operator  $G$  in  $\mathfrak{D}(G)$  mentioned above. In sections 3 to 5 the two different boundary spaces are investigated and the tools for the proof of Theorem 6.6 are prepared.

The author wishes to express his appreciation for stimulation and for many valuable discussions about this subject which he had with Professor R. S. Phillips during the first period he worked on the paper. Especially he is indebted to him for the knowledge of an example which suggested the principal idea of this paper.

**1. Auxiliary results.** In this paragraph we will establish some known results which have to be used essentially in the following.

LEMMA 1.1. *Let*

$$(1.1) \quad A(s) = A(s_1, \dots, s_p) = ((a_{ik}(s)))$$

*be a symmetric  $m \times m$ -matrix the coefficients  $a_{ik}(s)$  of which are continuously differentiable with respect to  $p$  variables  $s = (s_1, \dots, s_p)$  in a certain domain  $D_0$ . Let  $P_0(s)$ ,  $N_0(s)$ ,  $Z_0(s)$  be the orthogonal projections onto the subspaces of the  $m$ -component vector spaces which correspond to the eigenvalues  $\lambda_\nu(s) > 0$ ,  $\lambda_\nu(s) < 0$ ,  $\lambda_\nu(s) = 0$ , respectively.*

*Then the coefficients of  $P_0(s)$ ,  $N_0(s)$  and  $Z_0(s)$  are bounded measurable functions and the coefficients of*

$$(1.2) \quad P(s) = A(s)P_0(s), \quad N(s) = A(s)N_0(s)$$

*are Lipschitz continuous in any compact subregion of  $D_0$ . If*

$$(1.3) \quad u(s) = (u_1(s), u_2(s), \dots, u_m(s))$$

*is any bounded measurable  $m$ -component vector function defined in  $D_0$  for which*

$$(1.4) \quad v(s) = A(s)u(s)$$

is Lipschitz continuous in any compact subregion of  $D_0$ , then the same holds for

$$(1.5) \quad u_+(s) = P_0(s)u(s), \quad u_-(s) = N_0(s)u(s),$$

i.e.,  $u_+(s)$ ,  $u_-(s)$  are bounded measurable in  $D_0$  and  $v_z(s) = A(s)u_z(s)$  is Lipschitz continuous in any compact subregion of  $D_0$ .

*Proof.* It suffices to prove every statement for  $P_0(s)$ ,  $P(s)$  and  $v_+(s)$  only. Let  $D_1$  be any compact subregion of  $D_0$  and let  $\sigma$  be a positive number such that  $|A(s)u| \leq (\sigma - 1)|u|$  holds for every  $m$ -component vector  $u$  and for every  $s \in D_1$ . Here  $|u|^2 = \sum_{i=1}^m |u_i|^2$  with  $u_i$  being the components of  $u$ . Accordingly we denote  $\sum_{i=1}^m \bar{u}_i v_i$  by  $\bar{u}v$ . If  $R_z(s) = (A(s) - z)^{-1}$  denotes the resolvent of  $A(s)$  and if for  $0 \leq \epsilon < \sigma$   $C_\epsilon$  denotes the closed path in the complex  $z$ -plane which is the boundary of the region  $\epsilon \leq |z| \leq \sigma$ ,  $\operatorname{Re} z \geq 0$ , then it is well known that

$$(1.6) \quad P_0(s) = \frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0, \epsilon > 0} \int_{C_\epsilon} R_z(s) dz.$$

The integral has to be taken as a Cauchy mean value in case  $C_\epsilon$  crosses any eigenvalue of  $A(s)$ . On the other hand

$$\begin{aligned} R_{z_1}(s^1) - R_{z_2}(s^2) \\ &= R_{z_2}(s^2)(A(s^2) - z_2)R_{z_1}(s^1) - R_{z_2}(s^2)(A(s^1) - z_1)R_{z_1}(s^1) \\ &= R_{z_2}(s^2)\{A(s^2) - A(s^1) + (z_1 - z_2)\}R_{z_1}(s^1). \end{aligned}$$

This shows that for every  $z_0$  which is not an eigenvalue of  $A(s^0)$  the coefficients of  $R_z(s)$  are continuous and even continuously differentiable at  $(s^0, z_0)$ . Since  $A(s)$  is symmetric, all eigenvalues are real and obviously the path  $C_\epsilon$  does not contain any eigenvalue except perhaps at  $z = \epsilon$ . Especially we get

$$(1.7) \quad (\partial/\partial s_i)R_z(s) = -R_z(s)((\partial/\partial s_i)A(s))R_z(s).$$

This shows that  $P_0(s)$  is generated by a twice iterated limit process from a set of matrix functions  $R_z(s)$  having continuously differentiable coefficients. Consequently  $P_0(s)$  is measurable and, of course, bounded. Further it is well known that

$$(1.8) \quad P(s) = \frac{1}{2\pi i} \int_{C_0} R_z(s) A(s) dz$$

and

$$(1.9) \quad v_+(s) = \frac{1}{2\pi i} \int_{C_0} R_z(s) A(s) u(s) dz = \frac{1}{2\pi i} \int_{C_0} R_z(s) v(s) dz.$$

It suffices to prove the Lipschitz continuity of  $v_+(s)$ , since for  $u(s) = u_0 = \text{const.}$  the Lipschitz continuity of  $P(s)$  also follows.

For  $s^1, s^2 \in D_1$  and  $z \neq \lambda_\nu(s^1)$ ,  $z \neq \lambda_\nu(s^2)$ ,  $\nu = 1, \dots, m$ , we get

$$\begin{aligned} R_z(s^1)v(s^1) - R_z(s^2)v(s^2) \\ (1.10) \quad &= \{R_z(s^1) - R_z(s^2)\}v(s^1) + R_z(s^2)(v(s^1) - v(s^2)) \\ &= R_z(s^2)(A(s^2) - A(s^1))R_z(s^1)u(s^1) + R_z(s^2)(v(s^1) - v(s^2)). \end{aligned}$$

Hence

$$\begin{aligned} |v_+(s^1) - v_+(s^2)| \\ (1.11) \quad &\leq \frac{1}{2\pi} \left| \int_{\mathbb{G}_0} R_z(s^2)(A(s^2) - A(s^1))R_z(s^1)A(s^1)u(s^1)dz \right| \\ &\quad + \frac{1}{2\pi} \left| \int_{\mathbb{G}_0} R_z(s^2)(v(s^1) - v(s^2))dz \right|. \end{aligned}$$

For the second term we immediately get

$$\begin{aligned} (1.12) \quad \frac{1}{2\pi} \left| \int_{\mathbb{G}_0} R_z(s^2)(v(s^1) - v(s^2))dz \right| &\leq |v(s^1) - v(s^2)| \\ &\leq c |s^1 - s^2|, \end{aligned}$$

where

$$(1.13) \quad c = \sup_{s^1 \neq s^2; s^1, s^2 \in D_1} \{|v(s^1) - v(s^2)| |s^1 - s^2|^{-1}\}.$$

Here we used that  $\frac{1}{2\pi} \int_{\mathbb{G}_0} R_z(s^2)dz$  is a contraction matrix, as easily can be proved.

In order to estimate the first term let  $\phi^\nu, \lambda_\nu$ ,  $\nu = 1, \dots, m$ , and  $\psi^\nu, \mu_\nu$ ,  $\nu = 1, \dots, m$ , be the eigenvectors and eigenvalues of  $A(s^1)$  and  $A(s^2)$  respectively and denote

$$(1.14) \quad \overline{\psi}^\kappa (A(s^2) - A(s^1))\phi^\nu = q_{\kappa\nu}.$$

Clearly  $|q_{\kappa\nu}| \leq c |s^1 - s^2|$ ,  $s^1, s^2 \in D_1$ , where  $c$  is independent of  $s^1$  and  $s^2$ . Obviously then we get

$$\begin{aligned} R_z(s^1)A(s^1)u &= \sum_{\lambda_\nu \neq 0} \lambda_\nu (\lambda_\nu - z)^{-1} (\overline{\phi}^\nu u) \phi^\nu \\ (1.15) \quad R_z(s^2)u &= \sum_{\kappa=1}^m (\mu_\kappa - z)^{-1} (\psi^\kappa u) \psi^\kappa \end{aligned}$$

and thus

$$\begin{aligned} \frac{1}{2\pi} \left| \int_{\mathbb{G}_0} R_z(s^2)(A(s^2) - A(s^1))R_z(s^1)A(s^1)u(s^1)dz \right| \\ (1.16) \quad &= \frac{1}{2\pi} \left| \sum_{\lambda_\nu \neq 0} \sum_{\kappa=1}^m q_{\kappa\nu} (\overline{\phi}^\nu u) \psi^\kappa \int_{\mathbb{G}_0} \lambda_\nu [(\lambda_\nu - z)(\mu_\kappa - z)]^{-1} dz \right| \\ &\leq \frac{1}{2\pi} \sum_{\lambda_\nu \neq 0} \sum_{\kappa=1}^m |q_{\kappa\nu}| |\overline{\phi}^\nu u| \left| \int_{\mathbb{G}_0} \lambda_\nu [(\lambda_\nu - z)(\mu_\kappa - z)]^{-1} dz \right|. \end{aligned}$$

If  $\lambda_\nu = \mu_\kappa \neq 0$  then the corresponding integral in the above right hand side vanishes. But for  $\lambda_\nu \neq \mu_\kappa$ ,

$$(1.17) \quad \lambda_\nu [(\lambda_\nu - z)(\mu_\kappa - z)]^{-1} = \lambda_\nu (\mu_\kappa - \lambda_\nu)^{-1} [(\lambda_\nu - z)^{-1} - (\mu_\kappa - z)^{-1}].$$

Consequently

$$(1.18) \quad \int_{\mathbb{C}_0} \lambda_\nu [(\lambda_\nu - z)(\mu_\kappa - z)]^{-1} dz = 0$$

if  $\lambda_\nu$  and  $\mu_\kappa$  are both positive or both negative. Further we get

$$(1.19) \quad \left| \int_{\mathbb{C}_0} \lambda_\nu [(\lambda_\nu - z)(\mu_\kappa - z)]^{-1} dz \right| = 2\pi \left| \lambda_\nu (\mu_\kappa - \lambda_\nu)^{-1} \right|$$

if  $\lambda_\nu$  and  $\mu_\kappa$  have opposite sign and

$$(1.20) \quad \left| \int_{\mathbb{C}_0} \lambda_\nu [(\lambda_\nu - z)(\mu_\kappa - z)]^{-1} dz \right| = \pi \left| \lambda_\nu (\mu_\kappa - \lambda_\nu)^{-1} \right|$$

if  $\mu_\kappa = 0$ . But in both cases we also get  $\left| \lambda_\nu (\mu_\kappa - \lambda_\nu)^{-1} \right| \leq 1$ . Consequently

$$(1.21) \quad \begin{aligned} & \frac{1}{2\pi} \left| \int_{\mathbb{C}_0} R_z(s^2) (A(s^2) - A(s^1)) A(s^1) u(s^1) dz \right| \\ & \leq \sum_{\lambda_\nu \neq 0} \sum_{\kappa=1}^m |q_{\kappa\nu}| |\overline{\phi^\nu} u| \leq c |s^1 - s^2|. \end{aligned}$$

This proves the lemma.

**2. A special eigenvalue problem.** Let the operator  $L_1$  in  $\mathfrak{D}(L_1)$  be defined by

$$(2.1) \quad L_1 u = \sum_{i=1}^n a_i(x) \partial u / \partial x_i + b(x) u(x)$$

for  $u \in \mathfrak{D}(L_1)$ . The matrix functions  $a_i(x)$ ,  $b(x)$  are assumed to be  $m \times m$ -matrices defined and continuous in a domain  $D$  of  $(x_1, \dots, x_n)$ -space and on its boundary  $\Gamma$ . The matrices  $a_i(x)$  are assumed to be hermitian symmetric and with uniformly Hölder continuous first derivatives on  $D + \Gamma$ . Further the domain  $D$  is assumed to be bounded with its boundary  $\Gamma$  consisting of a finite number of simple nonintersecting hypersurfaces which are all twice Hölder continuously differentiable. We assume further the operator  $iL_1$  to be formally selfadjoint, i.e.,

$$(2.2) \quad b(x) + b^*(x) = \sum_{i=1}^n (\partial / \partial x_i) (a_i(x)), \quad x \in D + \Gamma,$$

where  $b^*(x)$  denotes the adjoint of the matrix  $b(x)$  (i.e., the transposed and complex conjugate matrix).

Let the domain  $\mathfrak{D}(L_1)$  be the space of all complex valued  $m$ -component vector functions defined in  $D + \Gamma$  satisfying the following conditions:

- a)  $u, \partial u / \partial x_i, i = 1, \dots, n$ , are continuous in  $D$ .
- b)  $u(x)$  is uniformly bounded in  $D$ .
- c)  $\lim_{\epsilon \rightarrow 0} u(x - \epsilon v) = u(x)$  holds for every  $x \in \Gamma$ , except possibly an  $(n-1)$ -dimensional null set.
- d)  $v(x) = A(x)u(x)$  is continuous on  $D + \Gamma$  and Lipschitz continuous on  $\Gamma$ .
- e)  $\int_D |L_1 u|^2 dx < \infty$ .

Let the  $m \times m$ -matrix  $A(x)$  be continuously differentiable and let its first derivatives satisfy a Hölder condition along the boundary  $\Gamma$ . Further let

$$(2.3) \quad A(x) = \sum_{i=1}^n a_i(x) v_i(x) \text{ on } \Gamma,$$

where  $v = (v_1(x), \dots, v_n(x))$  denotes the exterior normal on  $\Gamma$ .

Let  $B(x)$  be the positive square root of  $(A(x))^2$ :

$$(2.4) \quad B(x) \geq 0, \quad (B(x))^2 = (A(x))^2.$$

For the space  $\mathfrak{D}(L_1)$  we introduce the following two bilinear forms:

$$(2.5) \quad [u, v] = \int_{\Gamma} \overline{u(x)} B(x) v(x) d\sigma,$$

$$(2.6) \quad (u, v) = \int_D (\overline{L_1 u} L_1 v + \bar{u} v) dx.$$

Here  $d\sigma$  denotes the area element on  $\Gamma$ , and by  $\bar{z}w$  we mean the local inner product of the two complex valued  $m$ -component vectors  $z$  and  $w$ :

$$(2.7) \quad \bar{z}w = \sum_{i=1}^m \bar{z}_i w_i, \quad |z|^2 = \sum_{i=1}^m |z_i|^2.$$

Clearly both forms are positive:

$$(2.8) \quad [u, u] \geq 0, \quad (u, u) \geq 0, \quad u \in \mathfrak{D}(L_1).$$

In addition the form  $(u, u)$  is positive definite:  $(u, u) > 0, u \neq 0, u \in \mathfrak{D}(L_1)$ .

By adding ideal elements we complete the domain  $\mathfrak{D}(L_1)$  with respect to the positive definite form  $(u, u)$  to a Hilbert space which we call  $\mathfrak{H}^0$ . Since the metric

$$(2.9) \quad \|u\| = \{(u, u)\}^{\frac{1}{2}}$$

is stronger than that induced by the ordinary inner product in  $L_2$

$$(2.10) \quad \langle u, v \rangle = \int_D \bar{u}v \, dx,$$

the space  $\mathfrak{S}^0$  can be identified with a certain subspace of the space  $\mathfrak{S}^0$  of all (classes of equivalent) measurable  $m$ -component vector functions square integrable over  $D$ :

$$(2.11) \quad \langle \langle u \rangle \rangle^2 = \int_D |u|^2 \, dx < \infty.$$

So the ideal elements can be represented by square integrable functions too.

We still consider  $\mathfrak{D}(L_1)$  as a subspace of  $\mathfrak{S}$ , since by definition it is the domain of the operator  $L_1$ , which is an operator of  $\mathfrak{S}$  into itself. In order to discriminate between  $\mathfrak{D}(L_1)$  considered as a subspace of  $\mathfrak{S}^0$  and of  $\mathfrak{S}$  we denote the set of all elements of  $\mathfrak{D}(L_1)$  considered as a subspace of  $\mathfrak{S}^0$  by  $\mathfrak{S}^0(L_1)$ . This additional notation will prove to be necessary in the later considerations.

Our first aim is to prove the following

**THEOREM 2.1.** *There is a selfadjoint operator  $G^0$  defined in a dense subspace  $\mathfrak{D}(G^0)$  of  $\mathfrak{S}^0$  such that*

$$(2.12) \quad \mathfrak{D}(G^0) \supset \mathfrak{S}^0(L_1) \text{ and } [u, v] = (u, G^0v), \quad u, v \in \mathfrak{S}^0(L_1).$$

*Proof.* First note that by (2.2) and Green's formula the following relation holds:

$$(2.13) \quad \int_D (\bar{u}L_1v + \overline{L_1u}v) \, dx = \int_{\Gamma} \bar{u}Av \, d\sigma; \quad u, v \in \mathfrak{D}(L_1).$$

We denote the bilinear form introduced in (2.13) by  $Q(u, v)$ . By Schwartz' inequality

$$(2.14) \quad |Q(u, v)| \leq \|u\| \|v\|, \quad u, v \in \mathfrak{D}(L_1).$$

Consequently there exists a uniquely determined bounded linear selfadjoint operator  $Q^0$  defined in the whole space  $\mathfrak{S}^0$  such that

$$(2.15) \quad Q(u, v) = (Q^0u, v), \quad u, v \in \mathfrak{S}^0(L_1).$$

Now for any  $x \in \Gamma$  let

$$(2.16) \quad P(x) = \frac{1}{2}(B(x) + A(x)), \quad N(x) = \frac{1}{2}(B(x) - A(x)),$$

with the matrices  $A(x)$  and  $B(x)$  defined in (2.3) and (2.4). Denote the spaces of all  $v \in \mathfrak{S}^0(L_1)$  satisfying

$$(2.17) \quad P(x)v(x) = 0, \text{ or } N(x)v(x) = 0, \quad x \in \Gamma,$$

by  $\mathfrak{S}^{0-}$  and  $\mathfrak{S}^{0+}$  respectively. Then by (2.13)

$$(2.18) \quad [u, v] = \mp Q(u, v), \quad u \in \mathfrak{S}^{0+}, \quad v \in \mathfrak{S}^0(L_1).$$

For our theory it is of fundamental importance that

$$(2.19) \quad \mathfrak{S}^0(L_1) = \mathfrak{S}^{0+} + \mathfrak{S}^{0-},$$

where  $\mathfrak{S}^{0+} + \mathfrak{S}^{0-}$  denotes the space of all functions

$$(2.20) \quad u(x) = u_1(x) + u_2(x), \quad u_1(x) \in \mathfrak{S}^{0+}, \quad u_2(x) \in \mathfrak{S}^{0-}.$$

Clearly the inclusion  $\subset$  holds in (2.19).

In order to prove the inverse inclusion we first define two projection operators  $P_0(x)$  and  $N_0(x)$ . Let the symmetric  $m \times m$ -matrix  $P_0(x)$  project onto the subspace of all  $m$ -component vectors which is spanned by all eigenvectors corresponding to positive eigenvalues of  $A(x)$ . Accordingly let  $N_0(x)$  project onto the subspace spanned by the eigenvectors corresponding to negative eigenvalues. Given now any  $u(x) \in \mathfrak{S}^0(L_1)$  then by Lemma 1.1 the function  $u_0(x) = P_0(x)u(x)$ ,  $x \in \Gamma$ , is a bounded measurable  $m$ -component vector function defined for  $x \in \Gamma$ . Further by Lemma 1.1  $A(x)u_0(x) = P(x)u_0(x) = \frac{1}{2}(A(x)u(x) + B(x)u(x))$  is Lipschitz continuous on  $\Gamma$ . Now we apply the following theorem which has been proved by the author in [5].

**THEOREM 2.2.** *Let the domain  $D$ , its boundary  $\Gamma$ , the operator  $L_1$  in  $\mathfrak{D}(L_1)$  and the matrix  $A(x)$  satisfy all the assumptions mentioned above; especially let  $\mathfrak{D}(L_1)$  be characterized by the conditions a) to e). Then, if  $u_0(x)$  is any complex valued bounded measurable function defined on  $\Gamma$  and if*

$$v(x) = A(x)u_0(x)$$

*is Lipschitz continuous on  $\Gamma$ , there exists an  $m$ -component vector function  $u(x) \in \mathfrak{D}(L_1)$  such that  $u(x) = u_0(x)$ ,  $x \in \Gamma$ .*

Applying this theorem for our  $u_0(x)$  we immediately obtain the existence of some  $u_1(x) \in \mathfrak{D}(L_1)$  with  $u_1(x) = u_0(x)$  on  $\Gamma$ . We define  $u_2(x) = u(x) - u_1(x)$ ,  $x \in D + \Gamma$ . Clearly then  $u_2(x) = (1 - P_0(x))u(x)$ ,  $x \in \Gamma$ , and therefore  $N_0(x)u_2(x) = 0$ ,  $x \in \Gamma$ . Consequently (2.20) holds. This proves that

$$(2.21) \quad \mathfrak{D}(L_1) \subset \mathfrak{S}^{0+} + \mathfrak{S}^{0-}$$

and therefore that equation (2.19) is true.

One should observe that application of the complicated Theorem 2.2 can be avoided if the matrix  $A(x)$  is assumed to be of constant rank. This

follows because then, by a well known perturbation theorem of F. Rellich [16], the matrix  $P_0(x)$  has continuous first derivatives. Hence the vector function  $u_0(x)$  defined above also is Lipschitz continuous and its continuation into the interior of  $D + \Gamma$  to a function  $u_1(x) \in \mathfrak{D}(L_1)$  is trivial. In that case also one would be able to replace the domain  $\mathfrak{D}(L_1)$  defined above by the much simpler domain  $C^1(D + \Gamma)$  of all vector functions which are continuously differentiable on  $D + \Gamma$ , without disturbing any of the statements which follow in this paper.

Now for  $u \in \mathfrak{S}^{0,+}$ ,  $v \in \mathfrak{S}^0(L_1)$  we obtain the estimate

$$(2.22) \quad |[u, v]| = |Q(u, v)| \leq \|u\| \|v\|.$$

Therefore a vector function  $u^0 \in \mathfrak{S}^0$  exists such that

$$(2.23) \quad [u, v] = (u^0, v), \quad v \in \mathfrak{S}^0(L_1).$$

The correspondence  $u \rightarrow u^0$ , of course, is unique and linear and therefore defines a linear operator which is defined in  $\mathfrak{S}^{0,+}$ . Analogously by use of (2.17) for  $u \in \mathfrak{S}^{0,-}$ ,  $v \in \mathfrak{S}^0(L_1)$ , the same inequality (2.22) can be shown and therefore again a  $u^0 \in \mathfrak{S}^0$  exists such that (2.23) is true. This shows that for  $u \in \mathfrak{S}^{0,+} + \mathfrak{S}^{0,-} = \mathfrak{S}^0(L_1)$  there exists always an element  $u^0$  such that (2.23) holds. Simply decompose  $u(x)$  in the way indicated by (2.33) and define  $u^0 = u_1^0 + u_2^0$ . We define

$$(2.24) \quad G^0 u = u^0, \quad u \in \mathfrak{S}^0(L_1),$$

and then get a uniquely defined hermitian symmetric positive definite operator which satisfies

$$(2.25) \quad [u, v] = (G^0 u, v), \quad u, v \in \mathfrak{S}^0(L_1).$$

Now by a well known theorem of K. O. Friedrichs every hermitian symmetric positive definite operator has a selfadjoint extension. Let

$$(2.26) \quad ([u, v]) = (u, v) + [u, v]$$

and

$$(2.27) \quad ([u]) = ([u, u])^{\frac{1}{2}}.$$

Let further  $\mathfrak{P}^0$  be the completion of  $\mathfrak{S}^0(L_1)$  with respect to the positive definite metric  $([u])$ . Then by Friedrichs one of these extensions of  $G^0$  is given by the restriction of  $G^{0*}$  in  $\mathfrak{D}(G^{0*})$  to  $\mathfrak{P}^0 \cap \mathfrak{D}(G^{0*})$ . We chose this special extension to be our operator  $G^0$  and then we proved not only Theorem 2.1, but also the following.

Corollary to Theorem 2.1: the operator  $G^0$  in  $\mathfrak{D}(G^0)$  can be chosen in such a way that

$$(2.28) \quad \mathfrak{D}(G^0) \subset \mathfrak{P}^0, \quad [u, v] = (u, G^0 v), \quad u \in \mathfrak{P}^0, \quad v \in \mathfrak{D}(G^0).$$

Finally it is necessary to remark that for our theory it is very essential that already the operator  $G^{0'}$  in  $\mathfrak{D}(G^{0'}) = \mathfrak{S}^0(L_1)$  is essentially selfadjoint. This will be concluded in section 5 using a result about a special dissipative extension of  $L_0$  in  $\mathfrak{D}(L_0)$ , characterized by local boundary conditions, which was first obtained by K. O. Friedrichs [7], R. S. Phillips and P. D. Lax [15].

**3. The boundary space  $\mathfrak{B}$ .** Let  $\mathfrak{S}$  be the Hilbert space of all  $m$ -component vector functions  $u(x)$  which are square integrable over  $D$  and let  $\langle u, v \rangle$  denote the inner product as defined in (2.10). Let the operator  $L_0$  in  $\mathfrak{D}(L_0)$  be the restriction of  $L_1$  in  $\mathfrak{D}(L_1)$  to the space

$$(3.1) \quad \mathfrak{D}(L_0) = \{u(x) \mid u \in \mathfrak{D}(L_1), u = 0 \text{ outside a compact subset of } D\}.$$

Let  $L_0^*$  in  $\mathfrak{D}(L_0^*)$  be the (strict) adjoint of  $L_0$  in  $\mathfrak{D}(L_0)$ . By definition  $\mathfrak{D}(L_0^*)$  is the set of all  $u \in \mathfrak{S}$  for which an element  $u^* \in \mathfrak{S}$  exists such that  $\langle u, L_0 v \rangle = \langle u^*, v \rangle$ ,  $v \in \mathfrak{D}(L_0)$ , and then by definition  $L_0^* u = u^*$ . Because of (2.2) the operator  $iL_0$  certainly is hermitian symmetric and therefore we get  $L_0^* \supset -L_0$ ; i.e.,  $\mathfrak{D}(L_0^*) \supset \mathfrak{D}(L_0)$ ,  $L_0 u = -L_0 u$ ,  $u \in \mathfrak{D}(L_0)$ . More particularly we also get  $L_0^* \supset -L_1$  as (2.13) shows. Since  $L_0^*$  by definition is closed we get  $L_0^* \supset -L_1^{**}$ . It is a very essential fact for our theory that both of the operators of the last inclusion are equal:

$$(3.2) \quad L_0^* = -L_1^{**}.$$

This was proved first by K. O. Friedrichs [6]. Friedrichs calls the solutions  $u \in \mathfrak{D}(L_0^*)$  of  $-L_0^* u = f$  weak solutions, the solutions  $u \in \mathfrak{D}(L_1^{**})$  of  $L_1^{**} u = f$  strong solutions of the differential equation  $L_1 u = f$ . Using a certain type of integral operators, the so called mollifiers, he proves in his paper that every weak solution is also a strong solution, i.e., that  $L_0^* \subset -L_1^{**}$ . One can easily check that the assumptions for his theorem here are satisfied.

We now introduce the boundary space  $\mathfrak{B}$  by

$$(3.3) \quad \mathfrak{B} = \mathfrak{S}^0 \ominus \mathfrak{S}^0(L_0^{**}),$$

where the orthogonal complement is taken with respect to the inner product  $(u, v)$  defined in (2.6). Here  $\mathfrak{S}^0(L_0^{**})$  denotes the closed subspace of  $\mathfrak{S}^0$  which as a set of elements is equal to  $\mathfrak{D}(L_0^{**})$ . Using (3.8) it can easily be seen that the space  $\mathfrak{B}$  introduced by this definition is isomorphic to the boundary space  $\mathfrak{B}$  introduced by Calkin [2]. By (2.13) we obtain the relation

$$(3.4) \quad Q(u, v) = 0, \quad u \in \mathfrak{S}^0, \quad v \in \mathfrak{S}^0(L_0^{**}).$$

Since  $\mathfrak{S}^0(L_0^{**})$  and  $\mathfrak{B}$  by definition are closed under the norm  $\|u\|$ , every element  $\mathfrak{S}^0$  can be decomposed uniquely:

$$(3.5) \quad u = \phi + u_0, \quad \phi \in \mathfrak{B}, \quad u_0 \in \mathfrak{S}^0(L_0^{**}).$$

Therefore if  $u, v \in \mathfrak{S}^0$  and  $u = \phi + u_0$ ,  $v = \psi + v_0$  are the corresponding decompositions then by (3.3):

$$(3.6) \quad Q(u, v) = Q(\phi, \psi).$$

Hence the value of the form  $Q(u, v)$ ,  $u, v \in \mathfrak{S}^0$  is already determined by the orthogonal components of  $u$  and  $v$  in  $\mathfrak{B}$ . Therefore it suffices to consider  $Q(u, v)$  for  $u, v \in \mathfrak{B}$ .

The notation "boundary space" is justified by the following: If

$$u = \phi + u_0, \quad u_0 \in \mathfrak{S}^0(L_0^{**}), \quad \phi \in \mathfrak{B}, \quad v = \psi + v_0, \quad v_0 \in \mathfrak{S}^0(L_0^{**}),$$

then

$$(3.7) \quad u - v = u_0 - v_0 \in \mathfrak{D}(L_0^{**}).$$

The elements of  $\mathfrak{D}(L_0^{**})$  vanish at the boundary in some generalized sense. Therefore the function  $\phi$  determines the boundary values of  $u$  in some generalized manner. One may say that  $u$  and  $v$  have the same boundary values in this sense.

According to the theory developed by R. S. Phillips in [14] any closed operator  $M$  which is an extension of  $L_0^{**}$  and a contradiction of  $-L_0^*$  corresponds to a closed subspace

$$\mathfrak{B}(M) = \mathfrak{S}^0(M) \ominus \mathfrak{S}^0(L_0^{**}) \text{ of } \mathfrak{B}.$$

This correspondence is one to one. If  $M_1, M_2$  are two such operators then  $M_1 \subset M_2$  implies  $\mathfrak{B}(M_1) \subset \mathfrak{B}(M_2)$ . An operator  $M$  in  $\mathfrak{D}(M)$  is called dissipative if  $\langle u, Mu \rangle + \langle Mu, u \rangle \leq 0$ ,  $u \in \mathfrak{D}(M)$ . It is called maximal dissipative if it is dissipative and if it does not have any proper extension which is also dissipative.  $M$  is dissipative if and only if  $\mathfrak{B}(M)$  is negative with respect to the form  $Q(u, u)$ :

$$(3.8) \quad Q(u, u) \leq 0, \quad u \in \mathfrak{B}(M).$$

$M$  is maximal dissipative if and only if  $\mathfrak{B}(M)$  is maximal negative with respect to  $Q(u, u)$ , i.e., if every negative extension of  $\mathfrak{B}(M)$  coincides with  $\mathfrak{B}(M)$ .  $iM$  is selfadjoint if and only if  $\mathfrak{B}(M)$  is a nullspace of  $Q(u, u)$  which is maximal negative and maximal positive:

$$Q(u, u) = 0, \quad u \in \mathfrak{B}(M);$$

there are no proper extensions of  $\mathfrak{B}(M)$  which are entirely positive or entirely negative.

In order to get more information about the space  $\mathfrak{B}$  we remember that by definition  $\mathfrak{B}$  is the space of all  $\phi \in \mathfrak{S}^0$  such that

$$(3.9) \quad (\phi, u) = \langle \phi, u \rangle + \langle L_1^{**}\phi, L_0^{**}u \rangle = 0, \quad u \in \mathfrak{D}(L_0^{**}).$$

Hence

$$L_1^{**}\phi \in \mathfrak{D}(L_0^{**}) = \mathfrak{D}(L_1^{**})$$

and

$$(3.10) \quad L_1^{**}(L_1^{**}\phi) = \phi.$$

Here again (3.2) was used. Consequently  $\mathfrak{B}$  is the space of all solutions of

$$(3.11) \quad (L_1^{**})^2\phi = \phi.$$

Since  $\phi \in \mathfrak{B}$  implies  $\phi \in \mathfrak{D}(L_1^{**})$ , we obtain by applying  $L_1^{**}$  to (3.11) that

$$(3.12) \quad L_1^{**}\phi \in \mathfrak{B} \text{ for every } \phi \in \mathfrak{B}.$$

Therefore the operator  $L_1^{**}$  transforms the space  $\mathfrak{B}$  into itself. We denote the restriction of  $L_1^{**}$  to  $\mathfrak{B}$  by  $L$ ; then (3.11) implies

$$(3.13) \quad L^2 = 1.$$

On the other hand by (2.13) and (3.11) for  $u, v \in \mathfrak{B}$ :

$$(3.14) \quad \begin{aligned} (Lu, v) &= \langle Lu, v \rangle + \langle L^2u, Lv \rangle = \langle Lu, v \rangle + \langle u, Lv \rangle \\ &= Q(u, v) = (u, Lv). \end{aligned}$$

Further, if  $u, v \in \mathfrak{B}$ , then

$$(3.15) \quad (Lu, Lv) = \langle Lu, Lv \rangle + \langle L^2u, L^2v \rangle = \langle Lu, Lv \rangle + \langle u, v \rangle = (u, v).$$

Hence  $L$  is a hermitian symmetric and unitary operator of the space  $\mathfrak{B}$  into itself. In other words,  $L$  is a symmetry. Also we note that

$$(3.16) \quad Q(u, v) = (Lu, v), \quad u, v \in \mathfrak{B}.$$

Consequently  $L$  has eigenvalues at  $\lambda = +1$  and  $\lambda = -1$  at most. (For special operators  $L_1$  it may happen that one or both of the corresponding eigenspaces do not contain any element different from zero.) The identity

$$(3.17) \quad u = \frac{1}{2}(1 + L)u + \frac{1}{2}(1 - L)u$$

gives an eigenfunction expansion of the arbitrary element  $u \in \mathfrak{B}$ . The operators

$$(3.18) \quad \frac{1}{2}(1 + L), \quad \frac{1}{2}(1 - L)$$

are the projections onto the eigenspaces belonging to  $\lambda = +1$  and  $\lambda = -1$  respectively. Further we note that

$$(3.19) \quad Q(u, v) = \frac{1}{4}((1+L)u, (1+L)v) - \frac{1}{4}((1-L)u, (1-L)v)$$

for every  $u, v \in \mathfrak{B}$ . Using these facts Phillips also proves the following

**THEOREM 3.1.** *A negative subspace  $\mathfrak{B}(M)$  of  $\mathfrak{B}$  is maximal negative with respect to  $Q(u, u)$  if and only if*

$$(3.20) \quad (1-L)\mathfrak{B}(M) = (1-L)\mathfrak{B}.$$

We repeat the proof: First of all a maximal negative subspace must be closed under the norm of  $\mathfrak{B}$ , because otherwise the closure would be a proper negative extension. Since  $L$  has its spectrum only at  $\lambda = \pm 1$ , the space  $(1-L)\mathfrak{B}$  is closed and the space  $(1-L)\mathfrak{B}(M)$  must be closed also, because  $u^n \in \mathfrak{B}(M)$ ,  $(1-L)u^n \rightarrow v$  implies

$$\begin{aligned} 0 &\geq 4Q(\bar{u}^n - u^m, u^n - u^m) \\ &= \|(1+L)(u^n - u^m)\|^2 - \|(1-L)(u^n - u^m)\|^2 \end{aligned}$$

and therefore

$$(3.21) \quad \|(1+L)(u^n - u^m)\|^2 \leq \|(1-L)(u^n - u^m)\|^2 \rightarrow 0, \quad n, m \rightarrow \infty.$$

Hence

$$(3.22) \quad u_n = \frac{1}{2}(1+L)u_n + \frac{1}{2}(1-L)u_n$$

converges too. Let  $u = \lim_{n \rightarrow \infty} u_n$ ; then  $u$  is in the closure of  $\mathfrak{B}(M)$  and therefore in  $\mathfrak{B}(M)$ . Hence  $v = (1-L)u$  for an element  $u \in \mathfrak{B}(M)$ . Therefore if  $(1-L)\mathfrak{B}(M) \subset (1-L)\mathfrak{B}$  is a proper inclusion, the complement  $(1-L)\mathfrak{B} \ominus (1-L)\mathfrak{B}(M)$  will contain a vector  $\phi \neq 0$ . But then

$$Q(\phi + u, \phi + u) = Q(u, u) - \frac{1}{4}((1-L)\phi, (1-L)\phi) \leq 0, \quad u \in \mathfrak{B}(M).$$

Hence a proper negative extension exists and  $\mathfrak{B}(M)$  can not be a maximal negative.

Conversely, if (3.20) holds, then  $\mathfrak{B}(M)$  is maximal negative. For let  $\mathfrak{B}(M^0)$  be any negative extension of  $\mathfrak{B}(M)$  and let  $\phi \in \mathfrak{B}(M^0)$ . Then there exists a  $\psi \in \mathfrak{B}(M)$  such that  $(1-L)(\phi - \psi) = 0$ . But  $\phi - \psi \in \mathfrak{B}(M^0)$ ; hence

$$\begin{aligned} 0 &\geq 4Q(\phi - \psi, \phi - \psi) = \|(1+L)(\phi - \psi)\|^2 - \|(1-L)(\phi - \psi)\|^2 \\ &= \|(1+L)(\phi - \psi)\|^2. \end{aligned}$$

Consequently  $(1+L)(\phi - \psi) = 0$ ,  $\phi = \psi$ ,  $\mathfrak{B}(M^0) = \mathfrak{B}(M)$ . Therefore Theorem 3.1 is proved. Especially we also get the following

**COROLLARY.** *The closure of a negative subspace  $\mathfrak{B}(M)$  is maximal negative if and only if  $(1-L)\mathfrak{B}(M)$  is dense in  $(1-L)\mathfrak{B}$ .*

**4. The boundary space  $\mathfrak{E}$  and the space  $\mathfrak{B}$ .** When we consider the form  $[u, v]$  and the form  $Q(u, v)$  in their special expressions as boundary integrals then it seems to be more natural to consider both forms in another boundary space, which also would have more right to be called boundary space, namely in some space which consists of functions defined only on the boundary  $\Gamma$ .

Let  $A(x)$  and  $B(x)$  be the  $m \times m$ -matrices defined in (2.3) and (2.4). Let  $\mathfrak{E}''$  be the space of all (classes of equivalent) bounded measurable  $m$ -component vector functions  $u(x)$  defined for  $x \in \Gamma$  only for which  $A(x)u(x)$  is Lipschitz continuous on  $\Gamma$ , and let  $\mathfrak{B}''$  be the subspace of  $\mathfrak{E}''$  consisting of all those  $u \in \mathfrak{E}''$  for which  $A(x)u(x) = 0$  on  $\Gamma$ . Then by (2.5) and (2.13) the forms  $[u, v]$  and  $Q(u, v)$  can be defined for  $u, v \in \mathfrak{E}''$  and we get

$$(4.1) \quad [u, v] = 0, \quad Q(u, v) = 0, \quad \text{if } u \in \mathfrak{E}'', v \in \mathfrak{B}''.$$

It is easy to see that  $\mathfrak{B}''$  is just the nullspace of the form  $[u, u]$  in  $\mathfrak{E}''$ . Therefore, if we define

$$(4.2) \quad \mathfrak{E}' = \mathfrak{E}'' / \mathfrak{B}''$$

then the forms  $[u, v]$  and  $Q(u, v)$  induce two corresponding forms in  $\mathfrak{E}'$  and the value of  $[u, v]$  and  $Q(u, v)$  for  $u, v \in \mathfrak{E}''$  already is determined by the residue classes of  $u$  and  $v$  with respect to this factorization. Clearly the form  $[u, u]$  is positive definite in  $\mathfrak{E}'$ . We now complete  $\mathfrak{E}'$  with respect to the norm

$$(4.3) \quad [[u]] = \{[u, u]\}^{\frac{1}{2}}$$

and then call the completed space  $\mathfrak{E}$ .

We would like to compare this new boundary space with the boundary space  $\mathfrak{B}$ , defined in section 3. For this purpose let  $L_0'$  in  $\mathfrak{D}(L_0')$  be the restriction of  $L_1$  in  $\mathfrak{D}(L_1)$  to the space  $\mathfrak{D}(L_0')$  of all  $u \in \mathfrak{D}(L_1)$  which, restricted to the boundary, belong to the space  $\mathfrak{B}''$ . Clearly  $L_0 \subset L_0'$  and (2.13) yields  $L_0' \subset -L_1^* = L_0^{**}$ . On the other hand

$$(4.4) \quad \mathfrak{E}' \cong \mathfrak{D}(L_1) / \mathfrak{D}(L_0').$$

We can introduce the metric  $\|u\|$  for elements  $u \in \mathfrak{E}'$  by defining  $\|u\|$  to be equal to the greatest lower bound of the  $\|\cdot\|$ -norms of all elements of  $\mathfrak{D}(L_1)$  which belong to the residue class corresponding to  $u$ . By setting

$$(u, v) = \frac{1}{4} \{ \|u + v\|^2 - \|u - v\|^2 - i\|u + iv\|^2 + i\|u - iv\|^2 \}$$

we also get the form  $(u, v)$  defined for  $u, v \in \mathfrak{E}'$ . We prove

## LEMMA 4.1.

$$(4.5) \quad (u, u) > 0, \quad u \neq 0, \quad u \in \mathfrak{E}'.$$

In other words,  $(u, u)$  is positive definite also in  $\mathfrak{E}'$ .

*Proof.* Let  $(u, u) = \|u\|^2 = 0$  for some nonvanishing  $u \in \mathfrak{E}'$ . Then there exists a sequence  $u^n = u^0 - v^n$ ,  $n = 1, 2, \dots$ , such that  $u^0 \in \mathfrak{D}(L_1)$ ,  $u^0 \notin \mathfrak{D}(L_0')$ ,  $v^n \in \mathfrak{D}(L_1')$  and  $\lim_{n \rightarrow \infty} \|u^n\| = 0$ . Since  $Q(u, v) = 0$  for  $u \in \mathfrak{D}(L_0')$ ,  $v \in \mathfrak{D}(L_1)$ , and because the form is continuous with respect to the metric  $\|u\|$  this means that  $Q(u^0, v) = \lim_{n \rightarrow \infty} Q(u^n, v) = 0$  for every  $v \in \mathfrak{D}(L_1)$ . But by Theorem 2.2 there exists an element  $v^0 \in \mathfrak{D}(L_1)$  with  $v^0(x) = A(x)u^0(x)$ ,  $x \in \Gamma$ . The above limit for  $v = v^0$  and (2.13) furnish

$$Q(u^0, v^0) = \int_{\Gamma} |A(x)u^0(x)|^2 d\sigma = 0$$

or  $A(x)u^0(x) = 0$  a.e. on  $\Gamma$ . This means that  $u^0(x) \in \mathfrak{D}(L_0')$  which is a contradiction. Therefore Lemma 4.1 is proved.

The next conclusion is

## LEMMA 4.2.

$$(4.6) \quad \mathfrak{E}' \cong \mathfrak{D}(L_1)/\mathfrak{D}(L_0') \cong \mathfrak{S}^0(L_1) \ominus \mathfrak{S}^0(L_0^{**}),$$

where  $\mathfrak{S}^0(L_1)$  is defined as in Section 2, and where  $\mathfrak{S}^0(L_1) \ominus \mathfrak{S}^0(L_0^{**})$  means the set of the  $\mathfrak{B}$ -components of elements of  $\mathfrak{S}^0(L_1)$  according to the decomposition (3.5). The isomorphism includes the norm  $\|u\|$  and the form  $Q(u, v)$ .

*Proof.* Since  $\mathfrak{D}(L_0') \cong \mathfrak{S}^0(L_0')$  and  $\mathfrak{S}^0(L_0') \subset \mathfrak{S}^0(L_0^{**})$  all elements of a certain residue class of  $\mathfrak{D}(L_1)/\mathfrak{D}(L_0')$  have the same projection onto the space  $\mathfrak{S}^0(L_1) \ominus \mathfrak{S}^0(L_0^{**})$ . We establish the isomorphism between the above two spaces by assigning to each residue class  $u \in \mathfrak{D}(L_1)/\mathfrak{D}(L_0')$  the common projection of its elements onto the space  $\mathfrak{S}^0(L_1') \ominus \mathfrak{S}^0(L_0^{**})$ . Clearly this defines a homomorphism of  $\mathfrak{D}(L_1)/\mathfrak{D}(L_0')$  onto  $\mathfrak{S}^0(L_1) \ominus \mathfrak{S}^0(L_0^{**})$  which preserves the vector operations as well as the norm  $\|u\|$  and the form  $Q(u, v)$ . Hence we only have to show that the correspondence is one to one. Now, if  $u$  and  $v$  are different residue classes of  $\mathfrak{D}(L_1)/\mathfrak{D}(L_0')$ , then by Lemma 4.1 we get  $\|u - v\| \neq 0$ . But if  $u$  and  $v$  correspond to the same element of  $\mathfrak{S}^0(L_1) \ominus \mathfrak{S}^0(L_0^{**})$ , then  $u^0 - v^0 \in \mathfrak{D}(L_0^{**})$  for  $u^0 \in u$ ,  $v^0 \in v$ , and thus we will be able to find a sequence  $w^n \in \mathfrak{D}(L_0')$  with  $\lim_{n \rightarrow \infty} \|u^0 - v^0 - w^n\| = 0$ .

But this means that  $\|u - v\| = 0$  and therefore we get a contradiction.

For convenience we introduce the notation

$$(4.7) \quad \mathfrak{B}' = \mathfrak{S}^0(L_1) \ominus \mathfrak{S}^0(L_0^{**}).$$

Then Lemma 4.2 can be expressed in the form

$$(4.8) \quad \mathfrak{E}' \cong \mathfrak{B}'.$$

Now we can complete  $\mathfrak{E}'$  also with respect to the norm  $\|u\|$ . Since this completion obviously is isomorphic to the space  $\mathfrak{S}^0 \ominus \mathfrak{S}^0(L_0^{**}) = \mathfrak{B}$  we can state

**LEMMA 4.3.** *The completion of  $\mathfrak{E}'$  with respect to the metric  $\|u\|$  is isomorphic to the boundary space  $\mathfrak{B}$ , the isomorphism including the metric  $\|u\|$  and the form  $Q(u, v)$ .*

As a special fact we note that the isomorphism (4.8) introduces the form  $[u, v]$  also for the dense subspace  $\mathfrak{B}'$  of the boundary space  $\mathfrak{B}$ .

We find it convenient to introduce as a third positive definite inner product the form

$$(4.9) \quad ([u, v]) = (u, v) + [u, v], \quad u, v \in \mathfrak{E}' \cong \mathfrak{B}'.$$

We can complete the space  $\mathfrak{E}' \cong \mathfrak{B}'$  also with respect to the metric

$$(4.10) \quad ([u]) = \{([u, u])\}^{\frac{1}{2}}.$$

We denote this completion by  $\mathfrak{P}$  and its dense subspace corresponding to  $\mathfrak{E}' \cong \mathfrak{B}'$  by  $\mathfrak{P}'$ . When we consider the space  $\mathfrak{P}^0$  defined in Section 2, then we find between  $\mathfrak{P}$ ,  $\mathfrak{P}'$  and  $\mathfrak{P}^0$  the following relation. Clearly  $\mathfrak{P}^0 \supset \mathfrak{S}^0(L_0')$  and since  $[u, u] = 0$ ,  $u \in \mathfrak{S}^0(L_0')$ , we get  $([u, u]) = (u, u)$ ,  $u \in \mathfrak{S}^0(L_0')$ . This shows that there is a closed subspace  $\mathfrak{P}(L_0^{**})$  of  $\mathfrak{P}^0$  which corresponds elementwise to  $\mathfrak{S}^0(L_0^{**})$ . Now we state

**LEMMA 4.4.**

$$(4.11) \quad \mathfrak{P}' \cong \mathfrak{P}^0(L_1) \ominus \mathfrak{P}^0(L_0^{**}), \quad \mathfrak{P} \cong \mathfrak{P}^0 \ominus \mathfrak{P}^0(L_0^{**}),$$

where the orthogonal complement is taken with respect to the inner product  $([u, v])$ .

*Proof.* We only note that  $([u, v]) = (u, v)$  for  $u \in \mathfrak{P}^0$ ,  $v \in \mathfrak{P}^0(L_0^{**})$ . Therefore

$$\mathfrak{P}^0(L_1) \ominus \mathfrak{P}^0(L_0^{**}) \cong \mathfrak{S}^0(L_1) \ominus \mathfrak{S}^0(L_0^{**}) \cong \mathfrak{E}' \cong \mathfrak{B}' \cong \mathfrak{P}',$$

where the first orthogonal complement is taken with respect to the inner product  $([u, v])$  but the second with respect to the inner product  $(u, v)$ . Consequently the first formula (4.11) is proved and the second formula now

immediately follows from the fact that  $\mathfrak{P}$  and  $\mathfrak{P}^0$  are defined as completions of  $\mathfrak{P}'$  and  $\mathfrak{P}^0(L_1) \cong \mathfrak{S}^0(L_1)$  respectively. Especially we get the following

**COROLLARY.** *Orthogonality of  $u, v \in \mathfrak{P}^0$  with respect to  $(u, v)$  and with respect to  $([u, v])$  means the same if at least one of the elements  $u, v$  is contained in  $\mathfrak{P}^0(L_0^{**}) \cong \mathfrak{S}^0(L_0^{**})$ .*

**5. The operator  $G$  in  $\mathfrak{D}(G)$ .** Next we consider the form  $Q(u, v)$  in the space  $\mathfrak{E}$ . We refer to the definition of the spaces  $\mathfrak{S}_+^{0'}$  and  $\mathfrak{S}_-^{0'}$  in Section 2.  $\mathfrak{S}_+^{0'}$ , for instance, was the space of all  $u \in \mathfrak{S}^0(L_1)$  with

$$(5.1) \quad N(x)u(x) = \frac{1}{2}(B(x) - A(x))u(x) = 0 \text{ on } \Gamma,$$

and  $\mathfrak{S}_-^{0'}$ , the corresponding space with  $N(x)$  replaced by  $P(x)$ . We define

$$(5.2) \quad \mathfrak{E}_\pm' = \mathfrak{S}_\pm^{0'} \mathfrak{S}^0(L_0^1).$$

Clearly  $\mathfrak{E}' = \mathfrak{E}_+' \oplus \mathfrak{E}_-'$  under  $[u, v]$ . The relation (2.18) yields

$$(5.3) \quad [u, v] = \pm Q(u, v), \quad u \in \mathfrak{E}_\pm', \quad v \in \mathfrak{E}'.$$

If we denote the closures of  $\mathfrak{E}_+''$  and  $\mathfrak{E}_-'$  with respect to  $[[u]]$  by  $\mathfrak{E}_+$  and  $\mathfrak{E}_-$  respectively, then

$$(5.4) \quad \mathfrak{E} = \mathfrak{E}_+ \oplus \mathfrak{E}_- \text{ under } [u, v].$$

By (5.3) the form  $Q(u, v)$  can be extended continuously to all  $u \in \mathfrak{E}_+$ ,  $v \in \mathfrak{E}$ , and analogously to all  $u \in \mathfrak{E}_-$ ,  $v \in \mathfrak{E}$ . Since  $\mathfrak{E}_+$  and  $\mathfrak{E}_-$  are orthogonal complements of each other with respect to  $[u, v]$ , we can define a bounded operator  $Q$  in  $\mathfrak{E}$  by

$$(5.5) \quad Qu = \pm u, \quad u \in \mathfrak{E}_\pm.$$

We then get

**LEMMA 5.1.** *There exists a bounded selfadjoint unitary operator  $Q$  defined in all of  $\mathfrak{E}$  and having its spectrum only at  $\lambda = \pm 1$  at the most, satisfies the relation*

$$(5.6) \quad Q(u, v) = [Qu, v], \quad u, v \in \mathfrak{E}.$$

We now consider the operator  $G^0$  in  $\mathfrak{D}(G^0)$  defined in Theorem 2.1. First we again observe that  $\mathfrak{S}^0(L_0')$  and also its closure  $\mathfrak{S}^0(L_0^{**})$  are null-spaces of  $G^0$ . This follows because for  $\phi = \mathfrak{S}^0(L_0')$ ,  $v \in \mathfrak{S}^0(L_1)$ , Theorem 2.1 and formula (2.13) imply  $(G^0\phi, v) = [\phi, v] = 0$ .

Consequently the selfadjoint operator  $G^0$  in  $\mathfrak{D}(G^0)$  transforms the orthogonal complement  $\mathfrak{B}$  of  $\mathfrak{S}^0(L_0^{**})$  into itself. Hence the restriction of  $G^0$  to

$\mathfrak{D}(G) = \mathfrak{B} \cap \mathfrak{D}(G^0)$  considered as a transformation of  $\mathfrak{B}$  into itself is a selfadjoint operator. We call this operator  $G$  in  $\mathfrak{D}(G)$ .

LEMMA 5.4.

$$(5.7) \quad \mathfrak{B}' \subset \mathfrak{D}(G) \subset \mathfrak{B}.$$

*Proof.* By definition  $\mathfrak{B}' = \mathfrak{S}^0(L_1) \ominus \mathfrak{S}^0(L_0^{**})$ ,  $\mathfrak{D}(G) = \mathfrak{D}(G^0) - \mathfrak{S}^0(L_0^{**})$ , and  $\mathfrak{B} = \mathfrak{P}^0 \ominus \mathfrak{S}^0(L_0^{**})$ . But by Theorem 2.1 and its corollary,  $\mathfrak{S}^0(L_1) \subset \mathfrak{D}(G^0) \subset \mathfrak{P}^0$ . This proves the lemma.

LEMMA 5.5.

$$(5.8) \quad [u, v] = (u, Gv), \quad u \in \mathfrak{B}, \quad v \in \mathfrak{D}(G).$$

The proof is an obvious consequence of (2.28) and (5.7).

LEMMA 5.6.  $G$  in  $\mathfrak{D}(G) \subset \mathfrak{B}$  has a densely defined inverse.

*Proof.* Let  $G\phi = 0$ ,  $\phi \in \mathfrak{D}(G)$ . Then by (5.8),  $[\phi, v] = (G\phi, v) = 0$ ,  $v \in \mathfrak{B}$ . But  $\mathfrak{B}$  is dense in  $\mathfrak{E}$ , hence  $\phi = 0$ . Let  $G'$  in  $\mathfrak{D}(G')$  be the restriction of  $G$  in  $\mathfrak{D}(G)$  to  $\mathfrak{D}(G') = \mathfrak{B}'$ .

THEOREM 5.1. The operator  $G'$  in  $\mathfrak{D}(G')$  is essentially selfadjoint.

*Proof.* Let  $\phi$  be an element of  $\mathfrak{B}$  satisfying

$$(5.9) \quad (\phi, (G' + 1)u) = 0, \quad u \in \mathfrak{D}(G').$$

Then

$$(5.9a) \quad (\phi, (G^{o'} + 1)u) = 0, \quad u \in \mathfrak{D}(G^{o'}),$$

since  $\mathfrak{D}(G^{o'}) \subset \mathfrak{D}(G') + \mathfrak{S}^0(L_0^{**})$  and  $\mathfrak{S}^0(L_0^{**}) \perp \mathfrak{D}(G')$  under the inner product  $(u, v)$ . Since  $\mathfrak{D}(G^{o'}) = \mathfrak{S}^{0,+}_+ + \mathfrak{S}^{0,-}_-$ , (5.9a) holds for  $u \in \mathfrak{S}^{0,+}_+$  and  $u \in \mathfrak{S}^{0,-}_-$ . For  $u \in \mathfrak{S}^{0,+}_+$ , by definition  $(\phi, G'u) = Q(\phi, u) = \langle L\phi, u \rangle + \langle \phi, Lu \rangle$  and on the other hand  $(\phi, u) = \langle \phi, u \rangle + \langle L\phi, L_1u \rangle$ . Hence (5.9a), for  $u \in \mathfrak{S}^{0,+}_+$ , is equivalent to

$$\langle L\phi, u \rangle + \langle \phi, L_1u \rangle + \langle \phi, u \rangle + \langle L\phi, L_1u \rangle = 0$$

or

$$\langle (1 + L)\phi, (1 + L_1)u \rangle = 0, \quad u \in \mathfrak{S}^{0,+}_+.$$

Let  $L_+$  in  $\mathfrak{D}(L_+)$  be the restriction of  $L_1$  in  $\mathfrak{D}(L_1)$  to the space  $\mathfrak{D}(L_+) = \mathfrak{S}^{0,+}_+ \subset \mathfrak{D}(L_1)$ . Then it follows that

$$(5.10) \quad \langle (1 + L)\phi, (1 + L_+)u \rangle = 0, \quad u \in \mathfrak{D}(L_+).$$

Hence

$$(5.11) \quad (1 + L)\phi \in \mathfrak{D}(L_+^*), (1 + L_+^*)(1 + L)\phi = 0.$$

Now

$$(5.12) \quad L_+^* = -L_-^{**},$$

where  $L_-$  in  $\mathfrak{D}(L_-)$  is the restriction of  $L_1$  in  $\mathfrak{D}(L_1)$  to  $\mathfrak{D}(L_-) = \mathfrak{S}_-^{0'}$ . This follows from a theorem which first was proved by K. O. Friedrichs [6] and then was extended by R. S. Phillips and P. D. Lax [15] to the generality required here.\* Hence

$$(5.13) \quad (1 - L_-^{**})(1 + L)\phi = 0.$$

But for  $u \in \mathfrak{D}(L_-)$  we get

$$\begin{aligned} \langle (1 - L_-)u, u \rangle &= \langle u, u \rangle + \langle L_- u, L_- u \rangle - \langle L_- u, u \rangle - \langle u, L_- u \rangle \\ &= (u, u) - Q(u, u) \geq (u, u) \geq \langle u, u \rangle \end{aligned}$$

since  $Q(u, u) \leq 0$ ,  $u \in \mathfrak{D}(L_-) = \mathfrak{S}_-^{0'}$ . By closing the space  $\mathfrak{D}(L_-)$  we get

$$(5.14) \quad \langle (1 - L_-^{**})u, u \rangle \geq \langle u, u \rangle, \quad u \in \mathfrak{D}(L_-^{**}).$$

Here we substitute  $u = (1 + L)\phi$ . Then  $(1 - L_-^{**})u = 0$ . Hence

$$(5.15) \quad (1 + L)\phi = 0.$$

In the analogous manner we conclude that  $(1 - L)\phi = 0$ , using (5.9a) for  $u \in \mathfrak{D}(L_-) = \mathfrak{S}_-^{0'}$ . Hence

$$(5.16) \quad \phi = \frac{1}{2}(1 + L)\phi + \frac{1}{2}(1 - L)\phi = 0.$$

This shows that the inverse  $(1 + G')^{-1}$  is densely defined. Since on the other hand  $G'$  is positive we get that  $(1 + G')^{-1}$  is bounded. Hence  $G'$  in  $\mathfrak{D}(G')$  is essentially selfadjoint and the theorem is proved.

Next we state

**THEOREM 5.2.** *The operator  $G$  in  $\mathfrak{D}(G)$  satisfies the relation*

$$(5.17) \quad G^{-1} = LGL.$$

i.e.,

$$(5.18) \quad \begin{aligned} L\mathfrak{D}(G^{-1}) &= \mathfrak{D}(G) \\ G^{-1}u &= LGLu, \quad u \in \mathfrak{D}(G^{-1}). \end{aligned}$$

*Proof.* First we note that

$$(5.19) \quad Gu = \begin{cases} Lu, & u \in \mathfrak{E}_+^{0'} \\ -Lu, & u \in \mathfrak{E}_-^{0'}. \end{cases}$$

\* This paper still will not cover the case where the matrix  $A(x)$  defined in (2.3) changes rank on  $\Gamma$ . A paper of the author concerned with this case is in preparation.

This follows because for  $u \in \mathfrak{E}_+'$ ,  $v \in \mathfrak{E}'$ , the relation

$$(5.20) \quad (Gu, v) = [u, v] + \pm Q(u, v) = (\pm Lu, v)$$

holds. Now (5.19) and the relation  $L^2 = 1$  yield

$$(5.21) \quad G'^{-1} = \begin{cases} Lu, & Lu \in \mathfrak{E}_+' \\ -Lu, & Lu \in \mathfrak{E}_-'. \end{cases}$$

Replacing  $u$  by  $Lu$  in (5.21) we obtain

$$(5.22) \quad G'^{-1}Lu = \begin{cases} u, & u \in \mathfrak{E}_+' \\ -u, & u \in \mathfrak{E}_-'. \end{cases}$$

Hence

$$(5.23) \quad LG'^{-1}Lu = \begin{cases} Lu, & u \in \mathfrak{E}_+' \\ -Lu, & u \in \mathfrak{E}_-'. \end{cases}$$

and therefore

$$(5.24) \quad \begin{aligned} Lu &\in \mathfrak{D}(G^{-1}), \quad u \in \mathfrak{D}(G'), \\ LG'Lu &= G'u, \quad u \in \mathfrak{D}(G'). \end{aligned}$$

Now, given any  $u \in \mathfrak{D}(G)$  there exists a sequence  $u^n \in \mathfrak{D}(G')$  with  $u^n \rightarrow u$ ,  $Gu^n \rightarrow Gu$ , because by Theorem 5.1 the operator  $G'$  in  $\mathfrak{D}(G')$  is essentially selfadjoint. Hence  $LG^{-1}Lu^n = Gu^n \rightarrow Gu$ . Since  $L$  is bounded,  $Lu^n$  converges to  $Lu$ . Also  $G^{-1}Lu^n \rightarrow G^{-1}Lu$ . Consequently  $Lu \in \mathfrak{D}(G^{-1})$  and  $LG^{-1}Lu = Gu$ ,  $u \in \mathfrak{D}(G)$ . Hence  $LG^{-1}L = G$  and  $G^{-1} = LGL$  which proves Theorem 5.2.

Let  $E_\lambda$  be the spectral resolution of  $G$  in  $\mathfrak{D}(G)$ . Assume  $E_\lambda$  to be continuous from the right. Then set

$$(5.25) \quad F_\lambda = E_\lambda - E_{\lambda-0}, \quad \lambda > 1.$$

Obviously  $F_\lambda$  is a projection for every  $\lambda > 1$  and this projection is orthogonal under all three of the inner products  $(u, v)$ ,  $[u, v]$  and  $([u, v])$ .

**THEOREM 5.3.**  $F_\lambda$  commutes with  $L$ :

$$(5.26) \quad F_\lambda L = LF_\lambda, \quad \lambda > 1.$$

*Proof.* The relation  $G^{-1} = LGL$  yields  $f(G^{-1}) = Lf(G)L$ , i. e.,

$$\int_0^\infty f(\lambda^{-1}) dE_\lambda = L \left( \int_0^\infty f(\lambda) dE_\lambda \right) L$$

first for analytic functions, then, by passing to the limit, also for any arbitrary piecewise continuous function  $f(\lambda)$ . Now set

$$(5.27) \quad f_\mu(\lambda) = \begin{cases} 1, & \lambda \leq \mu \\ 0, & \text{elsewhere} \end{cases}$$

then

$$(5.28) \quad f_{\mu}(\lambda^{-1}) = \begin{cases} 1, & \lambda \geq \mu^{-1} \\ 0, & \text{elsewhere.} \end{cases}$$

Hence

$$(5.29) \quad f_{\mu}(G^{-1}) = \int_0^{\infty} f_{\mu}(\lambda^{-1}) dE_{\lambda} = \int_{\mu^{-1}-0}^{\infty} dE_{\lambda} = 1 - E_{\mu^{-1}-0}.$$

Consequently  $1 - E_{\mu^{-1}-0} = LE_{\mu}L$  or  $(1 - E_{\mu^{-1}-0})L = LE_{\mu}$ ,  $0 < \mu < \infty$ . Finally we observe that for  $\lambda > 1$ :  $F_{\lambda} = E_{\lambda}(1 - E_{\lambda^{-1}-0})$ . Hence

$$F_{\lambda}L = E_{\lambda}LE_{\lambda} = L(1 - E_{\lambda^{-1}-0})E_{\lambda} = LF_{\lambda}.$$

This proves the theorem.

**6. Boundary conditions in  $\mathfrak{E}$ .** Let  $P_0$  and  $N_0$  be the projections of  $\mathfrak{E}$  onto the closures  $\mathfrak{E}_+$  and  $\mathfrak{E}_-$  of  $\mathfrak{E}_+'$  and  $\mathfrak{E}_-'$  respectively. Obviously these projections are locally given by

$$(6.1) \quad P_0u = P_0(x)u(x), \quad N_0u = N_0(x)u(x),$$

where  $P_0(x)$  and  $N_0(x)$  are the matrices defined in Section 2. Now we state

**THEOREM 6.1.**

$$(6.2) \quad \begin{aligned} P_0u &= \frac{1}{2}(1 + G^{-1}L)u, & u \in \mathfrak{D}(G), \\ N_0u &= \frac{1}{2}(1 - G^{-1}L)u, & u \in \mathfrak{D}(G). \end{aligned}$$

This for  $u \in \mathfrak{D}(G')$  simply follows from (5.22). For  $u \in \mathfrak{D}(G)$  we simply obtain it by choosing a sequence  $u^n \in \mathfrak{D}(G')$  with  $u^n \rightarrow u$ ,  $Gu^n \rightarrow Gu$  under  $\|u\|$  and passing to the limit  $n \rightarrow \infty$ . Now let  $\mathfrak{P}_+$  and  $\mathfrak{P}_-$  be the closure of  $\mathfrak{E}_+'$  and  $\mathfrak{E}_-'$  under the norm of  $\mathfrak{P}$ .

**THEOREM 6.2.**

$$(6.3) \quad \mathfrak{D}(G) = \mathfrak{P}_+ + \mathfrak{P}_-.$$

*Proof.* Let  $u^n \in \mathfrak{E}_+'$ ,  $u^n \rightarrow u$  under the norm of  $\mathfrak{P}$ . By (5.21)  $u^n \in \mathfrak{E}_+'$  implies  $Gu^n = Lu^n \rightarrow Lu$ . Hence  $Gu^n$  converges also and therefore  $u \in \mathfrak{D}(G)$ . Hence  $\mathfrak{P}_+ \subset \mathfrak{D}(G)$  and analogously  $\mathfrak{P}_- \subset \mathfrak{D}(G)$ .

Finally let  $u \in \mathfrak{D}(G)$ ; then  $u = P_0u + N_0u$  and

$$(6.4) \quad \begin{aligned} P_0u &= \frac{1}{2}(1 + G^{-1}L)u \in \mathfrak{D}(G), \\ N_0u &= \frac{1}{2}(1 - G^{-1}L)u \in \mathfrak{D}(G). \end{aligned}$$

Hence

$$(6.5) \quad P_0u \in \mathfrak{P}_+, \quad N_0u \in \mathfrak{P}_-,$$

and (6.3) is proved.

Next we consider the operator  $F_\lambda$ ,  $\lambda > 1$ , defined in the preceding section.

THEOREM 6.3.

$$(6.6) \quad F_\lambda \mathfrak{E}_+ \subset \mathfrak{P}_+, \quad F_\lambda \mathfrak{E}_- \subset \mathfrak{P}_-.$$

*Proof.* Let  $\phi \in \mathfrak{E}_+'$ . Then  $GF_\lambda\phi = F_\lambda G\phi = F_\lambda L\phi = LF_\lambda\phi$ . Hence

$$N_0 F_\lambda \phi = \frac{1}{2}(1 - G^{-1}L)F_\lambda \phi = 0, \quad \phi \in \mathfrak{E}_+'.$$

Given any  $\phi \in \mathfrak{E}_+$  there exists a sequence  $\phi^n \in \mathfrak{E}_+'$  with  $\phi^n \rightarrow \phi$ ,  $n \rightarrow \infty$  under the norm of  $\mathfrak{E}$ . But  $F_\lambda$  is bounded under the norm of  $\mathfrak{E}$ . Hence  $F_\lambda \phi^n \rightarrow F_\lambda \phi$  under the norm of  $\mathfrak{E}$ . Since  $N_0 F_\lambda \phi^n = 0$ , it follows that  $N_0 F_\lambda \phi = 0$  which proves that  $F_\lambda \phi \in \mathfrak{E}_+$  for  $\phi \in \mathfrak{E}_+$ . Since on the other hand  $F_\lambda u \in \mathfrak{P}$  for every  $u \in \mathfrak{E}$  and every  $1 < \lambda < \infty$  we obtain  $F_\lambda \phi \in \mathfrak{P}_+$  and hence the desired inclusion. The second inclusion is proved in the analogous manner.

We will use now the operators  $P_0$  and  $N_0$  to define dissipative boundary conditions. This simply can be done as follows: Let  $I$  be any contraction operator mapping  $\mathfrak{E}_-$  into  $\mathfrak{E}_+$ :

$$(6.7) \quad \begin{aligned} Iu &\in \mathfrak{E}_+, & u &\in \mathfrak{E}_-, \\ [[Iu]] &\leq [[u]], & u &\in \mathfrak{E}_-. \end{aligned}$$

Then we can impose the condition

$$(6.8) \quad P_0 u = IN_0 u, \quad u \in \mathfrak{E}.$$

Let  $M(I)$  be the space of all  $u \in \mathfrak{P}$  satisfying (6.8). Using this space we define an operator  $M(I) = M$  in  $\mathfrak{D}(M)$  by the following prescription:

$$(6.9) \quad \begin{aligned} \mathfrak{D}(M) &= \{u \in \mathfrak{D}(L_1) \mid u = \phi + u_0, \quad u_0 \in \mathfrak{D}(L_0^{**}), \quad \phi \in \mathfrak{M}(I)\}, \\ Mu &= L_1^{**}u, \quad u \in \mathfrak{D}(M). \end{aligned}$$

We intend to show that, under a further condition,  $M$  in  $\mathfrak{D}(M)$  is essentially maximal dissipative, i. e., its closure is maximal dissipative. In order to prove this, we first mention that the operator  $M$  is dissipative:

$$\begin{aligned} \langle Mu, u \rangle + \langle u, Mu \rangle &= Q(u, u) \\ &= [u, Qu] = [P_0 u, P_0 u] - [N_0 u, N_0 u] \\ &= [[IN_0 u]]^2 - [[N_0 u]]^2 \leq 0 \end{aligned}$$

because of (6.7) and (6.8). Further, in order to investigate the maximality we use the Corollary of Theorem 3.1. Accordingly we have to try to prove that  $(1 - L)\mathfrak{M}(I)$  is dense in  $(1 - L)\mathfrak{P}$  under the norm of  $\mathfrak{P}$ . Now  $\mathfrak{M}(I)$

consists of all elements  $u \in \mathfrak{P}$  which are of the form  $u = N_0 v + IN_0 v$ ,  $v \in \mathfrak{E}$ . We define the operator  $W$  by

$$(6.10) \quad W = (1 - G)(1 + G)^{-1}.$$

Obviously  $W$  is a bounded selfadjoint operator under each of the three norms  $\|u\|$ ,  $[[u]]$  and  $([u])$ ; also  $W$  commutes with  $G$ . The norm of  $W$  is  $\leq 1$  in all three spaces  $\mathfrak{B}$ ,  $\mathfrak{E}$  and  $\mathfrak{P}$ .

#### THEOREM 6.4.

$$(6.11) \quad W\mathfrak{E}_+ \subset \mathfrak{E}_-, \quad W\mathfrak{E}_- \subset \mathfrak{E}_+.$$

*Proof.* To show this first inclusion let  $\phi = W\psi$ ,  $\psi \in \mathfrak{E}_+'$ ; then  $(1 + G)\phi = (1 - G)\psi$ . But by (5.21)  $G\psi = L\psi$ ,  $\psi \in \mathfrak{E}_+'$ . Hence  $(1 + G)\phi = (1 - L)\psi$ . Since  $\frac{1}{2}(1 - L)$  is a projection, we conclude that

$$\frac{1}{2}(1 - L)(1 + G)\phi = (1 - L)\psi = (1 + G)\phi.$$

But by Theorem 5.2 and Theorem 6.1

$$\begin{aligned} (1 - L)N_0\phi &= \frac{1}{2}(1 - L)(1 - G^{-1})\phi = \frac{1}{2}(1 - L - LG + G)\phi \\ &= \frac{1}{2}(1 - L)(1 + G)\phi. \end{aligned}$$

Hence  $(1 - L)N_0\phi = (1 + G)\phi$ . Again we use relation (5.40) and then get  $(1 + G)N_0\phi = (1 + G)\phi$ . Dividing by  $1 + G$  results in

$$(6.12) \quad N_0\phi = \phi, \quad \phi \in \mathfrak{E}_-.$$

By closing the space  $\mathfrak{E}_+'$  we get the same for every  $\psi \in \mathfrak{E}_+$ . In the analogous manner we prove the second inclusion.

Using Theorem 6.2 we see that the operator  $WI$  is a bounded transformation of  $\mathfrak{E}_-$  into itself; especially:

$$(6.13) \quad [[WIu]] \leq [[u]], \quad u \in \mathfrak{E}_-.$$

Actually it follows that

$$(6.14) \quad [[Wu]] < [[u]] \text{ for } u \neq 0, u \in \mathfrak{E}_-$$

and therefore in (6.13) the relation " $<$ " holds unless  $u = 0$ . Therefore certainly the inverse  $(1 + WI)^{-1}$  exists as an operator transforming a certain dense subspace of  $\mathfrak{E}_-$  into  $\mathfrak{E}_-$ . Let us assume that  $(1 + WI)^{-1}$  is bounded under the norm of  $\mathfrak{E}$ . This is certainly true if, for instance, the inequality (6.7) can be strengthened to

$$(6.14) \quad [[Iu]] \leq (1 - \epsilon)[[u]], \quad u \in \mathfrak{E}_-, \quad \epsilon > 0.$$

Denote the space  $(1 + WI)^{-1}\mathfrak{P}_-$  by  $\mathfrak{R}(I)$ . Clearly  $\mathfrak{R}(I) \subset \mathfrak{E}_-$ .

**THEOREM 6.5.** *If  $v \in \mathfrak{R}(I)$ , then*

$$(6.15) \quad u = N_0 v + IN_0 v = v + Iv \in \mathfrak{P}.$$

*Proof.* Let  $v \in \mathfrak{R}(I)$ . Then by definition of  $\mathfrak{R}(I)$

$$(6.16) \quad (1 + WI)v \in \mathfrak{P}_- \subset \mathfrak{D}(G).$$

Hence  $(1 - L)(1 + WI)v$  exists and by Theorem 6.1

$$(6.17) \quad (1 - L)(L + WI)v = (1 + G)(1 + WI)v.$$

Now set  $u = v + Iv$  and consider  $F_\lambda u$ ,  $\lambda > 1$ , with  $F_\lambda$  being the operator defined in (5.25). By definition of  $W$  and  $F_\lambda$  both operators commute. By (6.17) and Theorem 5.3 we obtain

$$(6.18) \quad F_\lambda(1 - L)(1 + WI)v = (1 - L)F_\lambda v + (1 - L)WF_\lambda Iv.$$

By Theorem 6.3  $F_\lambda v \in \mathfrak{P}_-$ ,  $F_\lambda I \in \mathfrak{P}_+$ . Hence by Theorem 6.4  $WF_\lambda Iv \in \mathfrak{P}_-$  and therefore by Theorem 6.1 and Theorem 6.2

$$(1 - L)WF_\lambda Iv = (1 + G)WF_\lambda Iv = (1 - G)F_\lambda Iv = (1 - L)F_\lambda Iv.$$

Hence (6.18) yields

$$(6.19) \quad F_\lambda(1 - L)(1 + WI)v = (1 - L)F_\lambda(v + Iv) = (1 - L)F_\lambda u.$$

Finally we observe that the following inequality holds:

$$(6.20) \quad \|w\|^2 \leq \frac{1}{2} \|(1 - L)w\|^2 + [[w]]^2, \quad w \in \mathfrak{P}.$$

To prove this we simply remark that for  $w \in \mathfrak{D}(G)$  the right hand side becomes

$$\begin{aligned} & \frac{1}{2} \|w\|^2 + \frac{1}{2} \|Lw\|^2 - \frac{1}{2} \{(w, Lw) + (Lw, w)\} + (w, Gw) \\ &= \|w\|^2 + (w, (G - L)w) \\ &= \|w\|^2 + 2[w, N_0 w] \\ &\geq \|w\|^2, \end{aligned}$$

since  $N_0$  is positive under  $[w, w]$ . For  $w \in \mathfrak{P}$  arbitrary we now simply choose a sequence  $w^n$  tending to  $w$  under  $([u])$  and then pass to the limit. Applying (6.20) for  $w = F_\lambda u$  we obtain

$$\begin{aligned} (6.21) \quad \int_{\lambda-1}^{\lambda} d\|E_\lambda u\|^2 &= \|F_\lambda u\|^2 \leq \frac{1}{2} \|(1 - L)F_\lambda u\|^2 + [[F_\lambda u]]^2 \\ &= \frac{1}{2} \|F_\lambda(1 - L)(1 + WI)v\|^2 + [[F_\lambda u]]^2. \end{aligned}$$

Now passing to the limit  $\lambda \rightarrow \infty$  the right hand side tends to

$$\frac{1}{2} \|(1-L)(1+WI)v\|^2 + [[u]]^2$$

which is a well defined number. Hence the integral  $\int_0^\infty d\|E_\lambda u\|^2$  exists and therefore  $u \in \mathfrak{B}$ . Since  $u \in \mathfrak{E}$  trivially holds we get  $u \in \mathfrak{P}$ , which proves Theorem 6.5. Now we will be able to prove the maximality of  $M$  in  $\mathfrak{D}(M)$ .

**THEOREM 6.6.** *If  $(1+WI)^{-1}$  is bounded under the norm of  $\mathfrak{E}$  then the operator  $M$  in  $\mathfrak{D}(M)$  is essentially maximal dissipative, i.e., its closure is maximal dissipative.*

*Proof.* We simply observe that by Theorem 6.5 every  $u = v + Iv$ ,  $v \in \mathfrak{R}(I)$ , is contained in  $\mathfrak{M}(I)$ . This is true because by Theorem 6.5  $u \in \mathfrak{P}$  and because  $v \in \mathfrak{R}(I) \subset \mathfrak{E}$ ,  $Iv \in \mathfrak{E}$ , and therefore

$$P_0 u = Iv = IN_0 u.$$

Now for any such  $u$  we get

$$\begin{aligned} F_\lambda(1-L)u &= (1-L)F_\lambda u = (1-L)F_\lambda v + (1-L)F_\lambda Iv \\ (6.22) \quad &= (1+G)F_\lambda v + (1-G)F_\lambda Iv \\ &= (1+G)F_\lambda(1+WI)v = (1-L)F_\lambda(1+WI)v. \end{aligned}$$

But by definition of  $\mathfrak{R}(I)$  we get  $(1+WI)\mathfrak{R}(I) = \mathfrak{P}_-$ . Hence  $(1+WI)v \in \mathfrak{P}_-$  and  $F_\lambda(1-L)u = F_\lambda(1-L)(1+WI)v$ . Passing to the limit  $\lambda \rightarrow \infty$  we obtain  $(1-L)u = (1-L)(1+WI)v$ . Hence the space  $(1-L)\mathfrak{M}(I)$  contains the space

$$(1-L)(1+WI)\mathfrak{R}(I) = (1-L)\mathfrak{P}_-.$$

Consequently we only have to show that  $(1-L)\mathfrak{P}_-$  is dense in  $(1-L)\mathfrak{B}$  under the norm of  $\mathfrak{B}$ . Now let  $Lf = -f$  and

$$(6.23) \quad (f, (1-L)u) = 0, \quad u \in \mathfrak{P}_-;$$

then also

$$(6.24) \quad (f, (1+L)u) = ((1+L)f, u) = 0, \quad u \in \mathfrak{P}_-.$$

Hence

$$(6.25) \quad (f, Lu) = 0, \quad u \in \mathfrak{P}_-.$$

Now replace  $u$  in (6.25) by  $F_\lambda u$ . This is legitimate because of Theorem 6.3. Hence by Theorem 5.3

$$(6.26) \quad 0 = (f, LF_\lambda u) = (f, F_\lambda Lu) = -(F_\lambda f, Gu) = -[F_\lambda f, u], \quad u \in \mathfrak{P}_-.$$

Consequently  $F_\lambda f \in \mathfrak{P}_+$ ,  $1 < \lambda < \infty$ . If  $\lambda$  tends to  $\infty$  then  $F_\lambda f$  tends to  $f$  under

the norm of  $\mathfrak{B}$ . But for  $v \in \mathfrak{B}_+$  we obtain the estimate

$$[v, v] = (v, Gv) = (v, Lv) \leq (v, v)$$

or  $[[v]] \leq \|v\|$ ,  $v \in \mathfrak{B}_+$ . Hence  $Ff$  converges also in  $\mathfrak{B}$  and therefore  $f \in \mathfrak{B}_+$ . But  $Lf = -f$  and therefore

$$0 \leq [f, f] = (f, Gf) = (f, Lf) = -(f, f) \leq 0.$$

Hence  $f = 0$  and the theorem is proved.

Finally it should be remarked that obviously the same calculations can be carried through for the operator  $-L_1$  instead of the operator  $L_1$  defined in (2.1). Then we obtain the same boundary spaces, etc., but  $\mathfrak{S}_+^{0'}$  and  $\mathfrak{S}_-^{0'}$ ,  $\mathfrak{E}_+$  and  $\mathfrak{E}_-$ , etc., get reversed. For the operators  $M$  we then get that  $-M$  is an essentially maximal positive continuation of  $L_0$ . Hence the preceding theory also can be applied for getting maximal positive operators  $M$ . This and Phillips' criteria mentioned in Section 3 will imply the following

**THEOREM.** *If  $I$  is a unitary transformation under the norm of  $\mathfrak{E}$  mapping  $\mathfrak{E}_-$  onto  $\mathfrak{E}_+$  and if  $(1 + WI)^{-1}$  is bounded under the norm of  $\mathfrak{E}$ , then the operator  $iM$  in  $\mathfrak{D}(M)$  is essentially selfadjoint.*

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# ON THE STIEFEL-WHITNEY CLASSES OF A MANIFOLD.\*<sup>1</sup>

By W. S. MASSEY.

**1. Introduction.** It has been well known for many years that various relations must hold between the Stiefel-Whitney classes of the tangent bundle of a manifold which do not hold for the Stiefel-Whitney classes of an arbitrary sphere bundle. For example, Whitney [6] showed that the 3-dimensional Stiefel-Whitney class of an orientable 4-manifold is always zero. The three main theorems of this paper are results of this kind. They assert that for certain integers  $n$  and  $k$ , the  $k$ -dimensional Stiefel-Whitney class (or dual Stiefel-Whitney class) of a compact  $n$ -manifold (or a compact orientable  $n$ -manifold) is always zero.

**2. Statement of results.** Throughout this paper we will use only the ring of integers mod 2,  $Z_2$ , for coefficients of any cohomology groups or cohomology classes considered. The notation  $M^n$  will be consistently used to denote a compact, connected,  $n$ -dimensional manifold,  $w_i \in H^i(M^n, Z_2)$  will denote the  $i$ -th Stiefel-Whitney class of its tangent bundle, and  $\bar{w}_i \in H^i(M^n, Z_2)$  will denote the  $i$ -th dual Stiefel-Whitney class. The Stiefel-Whitney classes and dual Stiefel-Whitney classes are related by the following formula:

$$(2.1) \quad \left( \sum_i w_i \right) \left( \sum_j \bar{w}_j \right) = 1.$$

According to the Whitney duality theorem, the  $\bar{w}_i$  are the Stiefel-Whitney classes of the normal bundle for any differentiable imbedding of  $M^n$  in a Euclidean space of any dimension.<sup>2</sup>

The following three elementary properties of the Stiefel-Whitney classes of an  $n$ -manifold  $M^n$  are well known (see Wu [7]):

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<sup>1</sup> During the preparation of this paper, the author was partially supported by a grant from the National Science Foundation. An abstract announcing the three main theorems of this paper was submitted to the American Mathematical Society in December, 1958 (see Notices Amer. Math. Soc., vol. 6, p. 143).

<sup>2</sup> In his thesis [5], R. Thom showed how to define the  $w_i$  and  $\bar{w}_i$ , when differentiability hypotheses are lacking.

- (2.2) If  $n$  is odd,  $w_n = 0$ .  
 (2.3)  $w_1 = 0$  if and only if  $M^n$  is orientable.  
 (2.4) For any  $n$ ,  $\bar{w}_n = 0$ .

Our theorems extend these results.

**THEOREM I.** *Let  $M^n$  be a compact,  $n$ -manifold and let  $q$  be an integer such that  $0 < q < n$ . If  $w_{n-q} \neq 0$ , then there exist integers  $h_1, \dots, h_q$  such that  $h_1 \geq h_2 \geq \dots \geq h_q \geq 0$  and*

$$n = 2^{h_1} + 2^{h_2} + \dots + 2^{h_q}.$$

Moreover, if  $M^n$  is orientable, the following additional restrictions must be imposed:

- (a)  $q \neq 1$ ,  
 (b) If  $n \equiv 2 \pmod{4}$ , then  $h_q \neq 1$ ,  
 (c) An odd number of the  $h_i$ 's are not equal to  $h_q + 1$ .

The proof of this theorem will be given in § 4. For the present, we will list the following corollaries.<sup>3</sup>

**COROLLARY 1.** *If  $\bar{w}_{n-1} \neq 0$ , then  $n$  is a power of 2 and  $M^n$  is non-orientable.*

This is the case  $q = 1$  of the theorem. Note that for  $n$ -dimensional real projective space one actually has  $\bar{w}_{n-1} \neq 0$  if  $n$  is a power of 2. To prove this, one can use the determination of the Stiefel-Whitney classes of  $n$ -dimensional real projective space by E. Stiefel [4] and formula (1) above.

**COROLLARY 2.** *If  $\bar{w}_{n-2} \neq 0$ , then  $n = 2^k(2^h + 1)$  for non-negative integers  $h$  and  $k$ . In addition, if  $M^n$  is orientable, the cases  $n = 2(2^h + 1)$  for  $h > 0$  and  $n = 3 \cdot 2^k$  are not possible.*

This is the case  $q = 2$  of the theorem.  $2^k(2^h + 1) = 2^{k+h} + 2^k$ , so let  $h_1 = k + h$  and  $h_2 = k$ . The two excluded cases correspond to cases (b) and (c) respectively of the main theorem.

**COROLLARY 3.** *If  $n = 2^r - 1$ , then  $\bar{w}_i = 0$  for  $i > n - r$ .*

*Proof.* Since  $2^r - 1 = 1 + 2 + 2^2 + \dots + 2^{r-1}$ , the minimum value of  $q$  which can occur in Theorem I is  $q = r$ .

<sup>3</sup> I have been informed by A. Shapiro that the result stated in corollary 1 has been obtained independently by A. Dold.

We leave it to the reader to derive other consequences of Theorem I. In doing this it is often useful to observe that the following two conditions are equivalent: (a)  $n = 2^{h_1} + 2^{h_2} + \cdots + 2^{h_q}$  for non-negative integers  $h_1, \cdots, h_q$ . (b) In the dyadic expansion of the integer  $n$ , the digit 1 does not occur more than  $q$  times.

**THEOREM II.** *If  $n$  is even and  $M^n$  is orientable, then  $w_{n-1} = 0$ .*

Wu indicates a proof of this result in case  $n \equiv 2 \pmod{4}$  (see [7]). The proof for the case  $n \equiv 0 \pmod{4}$  is given in § 5.

**THEOREM III.** *If  $n \equiv 3 \pmod{4}$  and  $M^n$  is orientable, then  $w_n = w_{n-1} = w_{n-2} = 0$ .*

This theorem is an easy consequence of Wu's formulas [7]. The proof is given in § 3.

In a certain sense, Theorems II and III together with statements (2.2) and (2.3) are the best that one can hope for in this direction. This may be seen by consideration of certain examples, as follows:<sup>4</sup>

In the case of non-orientable manifolds, statement (2.2) above is the best possible. For if  $n$  is even,  $n = 2k$ , then  $M^n = (P_2)^k$  (the Cartesian product of  $k$  copies of the real projective plane,  $P_2$ ) has non-vanishing Stiefel-Whitney classes in all dimensions, while if  $n$  is odd,  $n = 2k + 1$ , then  $M^n = (P_2)^k \times S^1$  (where  $S^1$  denotes a 1-sphere) has non-vanishing Stiefel-Whitney classes in all dimensions  $< n$ .

The case of orientable manifolds is more complicated. First consider the case where  $n = 4k$ . Let  $P_2(C)$  denote the complex projective plane (a 4-dimensional manifold), and let  $M^n = [P_2(C)]^k$ , the Cartesian product of  $k$  copies. Then  $w_i \neq 0$  for all even integers  $i \leq n$ ; in particular,  $w_{n-2} \neq 0$ , so Theorem II can not be improved if  $n = 4k$ . If  $n = 4k + 1$ , one may obtain examples of an  $M^n$  for which  $w_{n-1} \neq 0$  by taking  $M^n = [P_2(C)]^k \times S^1$ , or  $M^n = P(1, 2k)$ , a manifold considered by A. Dold in [1]. For  $n = 4k + 2$ , one may obtain an example of an  $M^n$  for which  $w_{n-2} \neq 0$  by taking  $M^n$

<sup>4</sup> For the proof of the assertions made in the following paragraphs about these example, the following result is needed. Let  $M$  and  $M'$  be compact manifolds. Identify the cohomology ring of the product space,  $H^*(M \times M', \mathbb{Z}_2)$ , with the tensor product  $H^*(M, \mathbb{Z}_2) \otimes H^*(M', \mathbb{Z}_2)$  as usual. If  $w = 1 + w_1 + w_2 + \cdots$  and  $w' = 1 + w'_1 + w'_2 + \cdots$  denote the total Stiefel-Whitney classes of  $M$  and  $M'$  respectively, then  $w \otimes w'$  is the total Stiefel-Whitney class of  $M \times M'$ . For the proof, see Thom, [5], pp. 142-143. One also needs to know that for the real projective plane,  $P_2$ ,  $w_1 \neq 0$  and  $w_2 \neq 0$  (see [4]); for the circle,  $S^1$ ,  $w_1 = 0$ ; and for the complex projective plane,  $P_2(C)$ ,  $w_2 \neq 0$  and  $w_4 \neq 0$ . The Stiefel-Whitney classes of Dold's manifolds  $P(m, n)$  have been computed by Dold [1].

$= [P_2(C)]^k \times S^1 \times S^1$  or  $M^n = P(1, 2k) \times S^1$ ; therefore Theorem II can not be improved in this case either. Similarly, for  $n = 4k + 3$  one obtains examples where  $w_{n-3} \neq 0$  by taking  $M^n = [P_2(C)]^k \times [S^1]^3$  or  $M^n = P(1, 2k) \times S^1 \times S^1$ . Thus Theorem III can not be improved. Whether or not Theorem I is a best possible theorem in this sense seems like a much more difficult question.

Of course there are other directions in which one could try to extend these theorems. For example, one could try to determine more general kinds of relations between Stiefel-Whitney classes of a manifold. An example is the relation  $w_1 w_2 = 0$  which holds in all manifolds of dimension  $\leq 5$  (see Wu [7]). This important problem seems very difficult, and outside of the case considered by A. Dold in [2], very little is known about it. One of the most pertinent problems in this connection is the following: Can any relation which holds between the Stiefel-Whitney classes of every  $n$ -manifold (or every orientable  $n$ -manifold) be derived from the formulas of Wu ([7] and [8])?

It should be pointed out that Theorem I may have implications for the problem of determining the lowest dimensional Euclidean space in which it is possible to imbed a given manifold. Whitney has proved that it is possible to imbed any  $n$ -dimensional smooth manifold differentiably in  $2n$ -dimensional Euclidean space. Moreover, if  $n = 2^k$ , then it is possible to give an example of an  $n$ -manifold which can not be imbedded in Euclidean  $(2n - 1)$ -space:  $n$ -dimensional real projective space  $P_n$  is such an example. To prove that  $P_n$  can not be imbedded in Euclidean  $(2n - 1)$ -space if  $n = 2^k$  one uses the fact that  $\bar{w}_{n-1} \neq 0$ . On the other hand, Corollary 1 of Theorem I shows that  $\bar{w}_{n-1} = 0$  for any  $n$ -manifold if  $n$  is not a power of 2. Thus it is natural to ask the following question: If  $n$  is not a power of 2, can any  $n$ -manifold be imbedded in Euclidean  $(2n - 1)$ -space? If the answer to this question is "no," it will require new methods to prove the existence of a counter-example.<sup>5</sup>

**3. Notation and preliminary results.** We will use the following notation and ideas in what follows. They are due to W. T. Wu [7].

(a)  $U_i \in H^i(M^n, Z_2)$  is the unique cohomology class such that

$$(3.1) \quad x \cdot U_i = Sq^i(x)$$

for any  $x \in H^{n-i}(M^n, Z_2)$ . The existence and uniqueness of the  $U_i$  follow from the Poincaré duality theorem. Note that  $U_0 = 1$ , and  $U_i = 0$  if  $i > \frac{1}{2}n$ .

<sup>5</sup> As a matter of fact, it was the search for examples of  $n$ -manifolds with  $w_{n-1} \neq 0$  which led the author to the discovery of Theorem I.

(b) The cohomology classes  $\bar{U}_i \in H^i(M^n, Z_2)$  are defined inductively by the equation

$$(3.2) \quad \left(\sum_i U_i\right) \cdot \left(\sum_i \bar{U}_i\right) = 1.$$

Here again  $\bar{U}_0 = 1$ . However it is *not* true in general that  $\bar{U}_i = 0$  for  $i > \frac{1}{2}n$ .

Wu proved the Stiefel-Whitney classes and dual Stiefel-Whitney classes may be expressed in terms of the  $U_i$  and  $\bar{U}_i$  respectively as follows:

$$(3.3) \quad w_k = \sum_i Sq^{k-i} U_i,$$

$$(3.4) \quad \bar{w}_k = \sum_i Sq^{k-i} \bar{U}_i.$$

These formulas are basic for all later computations.

In the following lemmas we record for later use some well known facts.

LEMMA 1. *A compact  $n$ -manifold,  $M^n$ , is orientable if and only if the homomorphism  $Sq^1: H^{n-1}(M^n, Z_2) \rightarrow H^n(M^n, Z_2)$  is trivial.*

This lemma is easily proved by using the fact that the homomorphism  $Sq^1$  is composition of the Bockstein homomorphism together with reduction mod 2, plus the known structure of the integral cohomology group in dimension  $n$  of an  $n$ -manifold.

LEMMA 2. *If  $M^n$  is orientable, then  $Sq^i: H^{n-i}(M^n, Z_2) \rightarrow H^n(M^n, Z_2)$  is zero for  $i$  odd.*

This follows from the known fact that  $Sq^i = Sq^1 Sq^{i-1}$  for  $i$  odd, together with Lemma 1.

LEMMA 3. *If  $M^n$  is orientable, then  $U_i = \bar{U}_i = 0$  for  $i$  odd.*

The fact that  $U_i = 0$  for  $i$  odd follows from Lemma 2 and the definition of  $U_i$ . Then one can prove that  $\bar{U}_i = 0$  for  $i$  odd by using formula (3.2).

In our proofs we need to make use of known properties of Steenrod squares and iterated Steenrod squares. For the sake of convenience, we will use the terminology and notation of Serre [3]. We assume the reader is familiar with the properties of Steenrod squares as listed in §2 of Serre's paper. Especially frequent use will be made of the properties of the homomorphism  $Sq^1$ . According to Cartan's formula,

$$(3.5) \quad Sq^1(x \cdot y) = (Sq^1 x) \cdot y + x \cdot (Sq^1 y),$$

i.e.,  $Sq^1$  is a derivation of the algebra  $H^*(X, Z_2)$ . In particular,

$$(3.6) \quad Sq^1(x^k) = kx^{k-1} \cdot Sq^1 x$$

for any positive integer  $k$ . Note also that

$$(3.7) \quad Sq^1 Sq^1 = 0.$$

This implies that for any odd integer  $i$ ,

$$(3.8) \quad Sq^1 Sq^i = 0.$$

We conclude this section by proving Theorem II. For an orientable manifold  $M^n$  of dimension  $n = 4k + 3$ ,  $U_i = 0$  unless  $i$  is even and  $0 \leq i \leq 2k$  (see Lemma 3). From this and (3.3) it follows that  $w_i = 0$  for  $i > 4k$  as desired.

**4. Proof of the Theorem I.** In the proof of Theorem I, frequent use will be made of the properties of iterated Steenrod squares. If  $I = (i_1, i_2, \dots, i_r)$  is any sequence of positive integers, then the notation  $Sq^I$  denotes the iterated Steenrod square  $Sq^{i_1} Sq^{i_2} \dots Sq^{i_r}$ . Such a sequence  $I = (i_1, i_2, \dots, i_r)$  is *admissible* if  $i_1 \geq 2i_2, i_2 \geq 2i_3, \dots, i_{r-1} \geq 2i_r$ . Every iterated Steenrod square may be expressed as a sum of admissible iterated Steenrod squares by repeated use of Adem's relations (see Serre [3], § 32).

With any admissible sequence of positive integers  $I = (i_1, i_2, \dots, i_r)$  one may associate a sequence of non-negative integers  $(\alpha_1, \alpha_2, \dots, \alpha_r)$  by the formulas

$$(4.1) \quad \alpha_1 = i_1 - 2i_2, \alpha_2 = i_2 - 2i_3, \dots, \alpha_{r-1} = i_{r-1} - 2i_r, \alpha_r = i_r.$$

It is clear that the sequence  $(\alpha_1, \dots, \alpha_r)$  determines without ambiguity the sequence  $(i_1, \dots, i_r)$ . The integer  $n(I) = i_1 + \dots + i_r$  is called the *degree* of  $I$ , and  $e(I) = \alpha_1 + \dots + \alpha_r$  is called the *excess* of  $I$ .

**LEMMA 4.** *For any mod 2 cohomology class  $x$ ,  $Sq^I(x) = 0$  if the degree of  $x$  is less than the excess of  $I$ .*

The proof depends on the fact that  $Sq^k(x) = 0$  if  $k$  is greater than the degree of  $x$ . The details are left to the reader.

**LEMMA 5.** *Let  $I = (i_1, \dots, i_r)$  be an admissible sequence of excess  $e(I)$ . Then there exists a unique admissible sequence  $J = (j_1, \dots, j_s)$  and a power of 2,  $m = 2^k$ , such that for any cohomology class  $x$  of degree  $e(I)$ ,*

$$Sq^I(x) = (Sq^J x)^m$$

*and  $e(J) < e(I)$ .*

For the proof, see the proof of Lemma 1, p. 204, of Serre [3].

Lemmas 4 and 5 together show that when considering iterated Steenrod squares operating on cohomology classes  $x$  of a fixed degree  $q$ , we can restrict our attention to those iterated squares  $Sq^I$  such that  $e(I) \leq q-1$ . In this case it is convenient (following Serre [3], p. 212) to let  $\alpha_0 = q-1-e(I)$ . Then one can derive the following formulae in case  $x$  is any mod 2 cohomology class of degree  $q$ :

$$\begin{aligned}
 \text{degree}(Sq^I x) &= n(I) + q \\
 (4.2) \quad &= \sum_{i=1}^r (2^i - 1)\alpha_i + q = \sum_{i=1}^r 2^i \alpha_i - e(I) + q \\
 &= \sum_{i=1}^r 2^i \alpha_i + \alpha_0 + 1 = 1 + \sum_{i=0}^r 2^i \alpha_i.
 \end{aligned}$$

Since  $\sum_{i=0}^r \alpha_i = \alpha_0 + e(I) = q-1$ , there are in all  $(q-1)$  powers of 2 in formula (4.2). Therefore we can rewrite this formula as follows,

$$(4.3) \quad \text{degree}(Sq^I x) = 1 + 2^{h_1} + 2^{h_2} + \cdots + 2^{h_{q-1}},$$

where  $h_1 \geq h_2 \geq \cdots \geq h_{q-1} \geq 0$ , and  $2^i$  occurs  $\alpha_i$  times in this sum (this is formula (17.5) of Serre [3], p. 212).

Next we will prove a couple of lemmas which are needed in the proof of Theorem I.

LEMMA 6. For any  $x \in H^k(M^n, Z_2)$ ,  $0 < k < n$ ,  $x \cdot \bar{w}_{n-k} = \sum_{i>0} (Sq^i x) \bar{w}_{n-k-i}$ .

*Proof.* By equation (3.4),

$$\begin{aligned}
 \bar{w}_{n-k} &= \sum_{i \geq 0} Sq^i \bar{U}_{n-k-i} \\
 &= \bar{U}_{n-k} + \sum_{i>0} Sq^i \bar{U}_{n-k-i}.
 \end{aligned}$$

By equation (3.2),

$$\bar{U}_{n-k} = \sum_{i>0} U_i \bar{U}_{n-k-i},$$

hence

$$\bar{w}_{n-k} = \sum_{i>0} (U_i \bar{U}_{n-k-i} + Sq^i \bar{U}_{n-k-i}).$$

Therefore if  $x \in H^k(M^n, Z_2)$ ,

$$x \cdot \bar{w}_{n-k} = \sum_{i>0} (x \cdot U_i \bar{U}_{n-k-i} + x \cdot Sq^i \bar{U}_{n-k-i}).$$

But

$$\begin{aligned}
 x U_i \bar{U}_{n-k-i} &= U_i (x \bar{U}_{n-k-i}) = Sq^i x \cdot \bar{U}_{n-k-i} \\
 &= \sum_{r=0}^i (Sq^r x) (Sq^{i-r} \bar{U}_{n-k-i}),
 \end{aligned}$$

from which it follows that

$$\begin{aligned} x \cdot \bar{w}_{n-k} &= \sum_{i \geq 0} \sum_{r=1}^i (Sq^r x) (Sq^{i-r} \bar{U}_{n-k-i}) \\ &= \sum_{0 < r \leq i} (Sq^r x) (Sq^{i-r} \bar{U}_{n-k-i}) \\ &= \sum_{r \geq 0} [(Sq^r x) \sum_{j \geq 0} Sq^j \bar{U}_{n-k-r-j}] \\ &= \sum_{r \geq 0} (Sq^r x) \bar{w}_{n-k-r}, \end{aligned}$$

as was to be proved.

LEMMA 7. The homomorphism  $H^k(M^n, Z_2) \rightarrow H^n(M^n, Z_2)$  defined by  $x \rightarrow x \cdot \bar{w}_{n-k}$  is a sum of iterated Steenrod squares.

In view of Lemma 6, this lemma is obvious: one applies Lemma 6 repeatedly until the desired reduction to a sum of iterated Steenrod squares is obtained.

We are now in a position to prove Theorem I. Assume that  $\bar{w}_{n-q} \in H^{n-q}(M^n, Z_2)$  is non-zero. By the Poincaré duality theorem, the homomorphism  $H^q(M^n, Z_2) \rightarrow H^n(M^n, Z_2)$  defined by  $x \rightarrow x \cdot \bar{w}_{n-q}$  is also non-zero. By Lemma 7, this homomorphism is a sum of iterated Steenrod squares, which we may assume to be admissible on account of Adem's relations. Hence the hypothesis of the theorem implies the following statement: There exists a non-zero admissible iterated Steenrod square

$$Sq^I: H^q(M^n, Z_2) \rightarrow H^n(M^n, Z_2),$$

where  $I = (i_1, \dots, i_r)$ . By Lemma 4,  $e(I) \leq q$ . Moreover, if  $e(I) = q$ , it follows from Lemma 5 that there exists an admissible sequence  $J = (j_1, \dots, j_s)$  and a power of 2,  $m = 2^k$ , such that

$$Sq^I(x) = [Sq^J(x)]^m$$

and  $e(J) < q$ . Therefore

$$\begin{aligned} n &= \text{degree}(Sq^I x) = 2^k \cdot \text{degree}(Sq^J x) \\ (4.4) \quad &= 2^k(2^{k_1} + 2^{k_2} + \dots + 2^{k_{q-1}} + 1) \end{aligned}$$

by equation (4.3). Here  $k_1, k_2, \dots$  are integers such that  $k_1 \geq k_2 \geq \dots \geq k_{q-1} \geq 0$ . If now we let

$$(4.5) \quad h_1 = k_1 + k, h_2 = k_2 + k, \dots, h_{q-1} = k_{q-1} + k, h_q = k,$$

then (4.4) takes the form

$$(4.6) \quad n = 2^{h_1} + 2^{h_2} + \cdots + 2^{h_q}$$

with  $h_1 \geq h_2 \geq \cdots \geq h_q \geq 0$ , and the first part of the theorem is proved.

Next, we will assume that  $M^n$  is orientable and prove the remaining parts of the theorem. In this case we can apply the results of Lemmas 1 and 2.

First assume that  $q = 1$ . Then  $n = 2^{h_1}$  from what we have just proved, and  $h_1 = k$  according to equation (4.5). Therefore the only non-zero iterated Steenrod square

$$Sq^I: H^1(M^n, Z_2) \rightarrow H^n(M^n, Z_2)$$

would be of the form  $Sq^I(x) = x^n$  with  $n = 2^k$ . Since  $n$  is even,  $x^n = Sq^1(x^{n-1})$  by equation (3.6). By use of Lemma 1, we see that if  $x^n \neq 0$ , then  $M^n$  is non-orientable, as was to be proved.

Next we will consider the case where  $n \equiv 2 \pmod{4}$ , i.e.,  $n = 4l + 2$ , and  $h_q = 1$ . Then it follows from (4.5) that  $k = 1$ . Therefore  $Sq^I(x) = [Sq^J(x)]^2$ ; and by equation (4.4),

$$n = \text{degree}(Sq^I x) = 2 \cdot \text{degree}(Sq^J x).$$

Hence  $\text{degree}(Sq^J x) = n/2 = 2l + 1$ . Thus

$$Sq^I(x) = Sq^{2l+1}[Sq^J(x)]$$

which is zero by Lemma 2. But this is a contradiction. Thus part (b) of Theorem I is proved.

Finally, we consider the case where an odd number of the  $h_i$ 's are equal to  $h_q + 1$ . Then it follows from equation (4.5) that an odd number of the  $k_i$ 's are equal to 1. Thus in (4.3), the summand  $2^1$  occurs an odd number of times,<sup>6</sup> i.e.,  $\alpha_1$  is odd, it follows from equation (4.1) that  $j_1$  is odd in the expression

$$Sq^I(x) = [Sq^J(x)]^m = [Sq^{j_1} \cdots Sq^{j_s}(x)]^m,$$

where  $m = 2^k$ . Since  $j_1$  is odd,

$$Sq^{j_1} = Sq^1 Sq^{j_1-1},$$

and  $Sq^1 Sq^J(x) = 0$  by equation (3.8). Therefore

$$[Sq^J(x)]^m = Sq^1 \{ [Sq^{j_1-1} Sq^{j_2} \cdots Sq^{j_s}(x)] \cdot [Sq^J(x)]^{m-1} \}$$

which is again zero by Lemma 1. Thus we have again reached a contradiction, and part (c) is proved.

<sup>6</sup> Actually, we are here concerned with the analog of equation (4.3) which is obtained by replacing  $I$  by  $J$  and  $h_i$  by  $k_i$  for  $i = 1, 2, \dots, q-1$ .

**5. Proof of Theorem II for the case  $n \equiv 0 \pmod{4}$ .** The following well-known lemma will be used in the course of the proof:

**LEMMA 8.** *If  $x$  is a mod 2 cohomology class of degree 1, then*

$$Sq^j(x^k) = C_j^k x^{k+j},$$

where  $C_j^k$  is the binomial coefficient reduced mod 2. In particular, if  $k$  is a power of 2, then  $Sq^j x^k = 0$  unless  $j = 0$  or  $j = k$ .

The proof is left to the reader.

Now assume that  $M^n$  is a compact orientable manifold of dimension  $n = 4k$ . Then

$$w_{n-1} = w_{4k-1} = Sq^{2k-1} U_{2k}$$

by (3.3). To prove that  $w_{n-1} = 0$ , it suffices to prove that  $x \cdot w_{n-1} = 0$  for any  $x \in H^1(M^n, Z)$ . Now

$$\begin{aligned} x \cdot w_{n-1} &= x \cdot Sq^{2k-1} U_{2k} \\ &= Sq^{2k-1}(x \cdot U_{2k}) + (Sq^1 x)(Sq^{2k-2} U_{2k}). \end{aligned}$$

However the first term on the right is zero by Lemma 2, and in the second term,  $Sq^1 x = x^2$ . Therefore

$$(5.1) \quad x \cdot w_{n-1} = x^2 \cdot Sq^{2k-2} U_{2k}$$

We will now show that if  $p = 2^q$  is a power of 2 and  $2 \leq p < 2k$ , then

$$(5.2) \quad x^p \cdot Sq^{2k-p} U_{2k} = x^{2p} \cdot Sq^{2k-2p} U_{2k}.$$

To prove this, one computes as follows:

$$(5.3) \quad x^p Sq^{2k-p} U_{2k} = Sq^{2k-p}(x^p \cdot U_{2k}) + x^{2p} Sq^{2k-2p} U_{2k}.$$

Here we have used the formula for the Steenrod square of a cup product together with Lemma 8. Next, note that

$$\begin{aligned} Sq^{2k-p}(x^p U_{2k}) &= U_{2k-p}(x^p U_{2k}) \\ (5.4) \quad &= U_{2k}(x^p U_{2k-p}) = Sq^{2k}(x^p U_{2k-p}) \\ &= (x^p U_{2k-p})^2 = x^{2p} \cdot U_{2k-p}^2 \\ &= Sq^1(x^{2p-1} \cdot U_{2k-p}^2) = 0 \end{aligned}$$

by (3.5), (3.6), and Lemma 1. Substitution of (5.4) in (5.3) gives (5.2), as desired.

One can now apply (5.2) to (5.1) repeatedly with  $p = 2, 4, 8, \dots$ , in

succession. If  $n$  is not a power of 2, this procedure leads to the result that  $x \cdot w_{n-1} = 0$ , as desired. If  $n$  is a power of 2, this same procedure shows that  $x \cdot w_{n-1} = x^n$  for any  $x \in H^1(M^n, \mathbb{Z})$ . However in this case, since  $n$  is even,

$$x^n = Sq^1(x^{n-1})$$

by (3.6). But  $Sq^1(x^{n-1}) = 0$  by Lemma 1, as was to be proved. The proof of Theorem II is complete.

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# ON A SUBALGEBRA OF $L(-\infty, \infty)$ .\*

By JOHN WERMER.

Let  $L$  denote the group algebra of the real line, i.e. the algebra under convolution of summable functions on  $(-\infty, \infty)$  with norm defined by

$$\|f\| = \int_{-\infty}^{\infty} |f(x)| dx.$$

Let  $L^*$  be the closed subalgebra of  $L$  consisting of all functions in  $L$  which vanish on the negative half-line.

In his paper "On the Maximality of Vanishing Algebras," [3], A. B. Simon refers to the following result:

**THEOREM.**  $L^*$  is a maximal closed subalgebra of  $L$ .

We now give a proof of this theorem, by deducing it from the following:

**LEMMA.** Let  $B$  be a commutative semi-simple Banach algebra with maximal ideal space the unit circle  $|\lambda| = 1$ , hence a function-algebra on  $|\lambda| = 1$ . Assume:

(1) The functions  $\lambda$  and  $1/\lambda$  lie in  $B$  and generate a dense subalgebra of  $B$ .

Let  $B^+$  be the algebra of those functions in  $B$  which have continuous extensions to  $|\lambda| \leq 1$  which are analytic in  $|\lambda| < 1$ . Then  $B^+$  is a maximal subalgebra of  $B$ .

*Proof.*  $B^+$  contains all powers of  $\lambda$  with non-negative exponent. Let  $Q$  be any closed proper subalgebra of  $B$  containing  $B^+$ . Then  $1/\lambda \notin Q$ , for else  $\lambda^n \in Q$  for all integers  $n$ , whence  $Q = B$  by (1). Hence there exists a multiplicative linear functional  $\chi$  on the algebra  $Q$  with  $\chi(\lambda) = 0$ . For all  $f$  in  $Q$ , then

$$|\chi(f)| \leq \lim_{n \rightarrow \infty} \|f^n\|_Q^{1/n} = \lim_{n \rightarrow \infty} \|f^n\|_{B^+}^{1/n} = \max_{|\lambda|=1} |f(\lambda)|.$$

By the Riesz representation theorem, there exists a measure  $d\mu$  on  $|\lambda| = 1$ ,

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$d\mu \neq 0$ , with

$$\chi(f) = \int_{|\lambda|=1} f(\lambda) d\mu(\lambda), \text{ all } f \text{ in } Q.$$

In particular, if  $g \in Q$ ,  $\lambda^n \cdot g^m \cdot \lambda \in Q$  if  $n \geq 0$ ,  $m \geq 0$ , and  $\chi(\lambda^n g^m \lambda) = 0$ , since  $\chi(\lambda) = 0$ , whence

$$0 = \int_{|\lambda|=1} g^m(\lambda) \lambda^n d\sigma(\lambda), n, m \geq 0,$$

where  $d\sigma(\lambda) = \lambda d\mu(\lambda) \neq 0$ . It follows from this, by a theorem given in [1], that  $g$  admits an analytic extension to  $|\lambda| < 1$ . Since this holds for all  $g$  in  $Q$ ,  $Q = B^+$ . This proves the Lemma.

*Note.* This Lemma was suggested to the author by the reasoning of Singer and Hoffman, in their paper, [2]. The same Lemma was independently noticed by DeLeeuw.

*Proof of Theorem.* We prove the maximality of  $L^+$  by introducing a related algebra  $B$  on the unit circle which satisfies the conditions of the Lemma. To each  $f \in L$  assign  $f^*$  on  $|\lambda| = 1$  defined by

$$f^*(\lambda) = \int_{-\infty}^{\infty} f(x) \exp(-x(1+\lambda)/(1-\lambda)) dx.$$

Put  $B = \{f^* + c \mid f \in L, c \text{ a constant}\}$  and norm  $B$  by setting  $\|f^* + c\|_B = \|f\|_L + |c|$ . Then  $B$  is a commutative semi-simple Banach algebra, since  $L$  is, and also  $1 \in B$ . Also the maximal ideal space of  $B$  is the unit circle. We claim  $\lambda$  and  $1/\lambda \in B$ . For let  $f_1(x) = 0$ ,  $x < 0$ ,  $f_1(x) = 2e^{-x}$ ,  $x \geq 0$ . Then  $f_1 \in L$  and so  $f_1^* \in B$ . But  $f_1^* = 1 - \lambda$  and so  $\lambda \in B$ . Similarly,  $1/\lambda \in B$ . We next claim  $\{\lambda^n \mid -\infty < n < \infty\}$  span  $B$ . For let  $\Phi$  be any bounded linear functional on  $B$ . By the well-known representation of such functionals on  $L$ , there exists a bounded function  $\phi$  on  $(-\infty, \infty)$  with

$$\Phi(f^*) = \int_{-\infty}^{\infty} f(x) \phi(x) dx, \text{ all } f \in L.$$

Let  $f_m$  be the  $m$ -fold convolution of  $f_1$ . Then  $f_m(x) = 0$ ,  $x < 0$ ,  $f_m(x) = c_m x^{m-1} e^{-x}$ ,  $x \geq 0$ ,  $c_m$  a constant. Let  $\Phi$  be a bounded linear functional on  $B$  with  $\Phi(\lambda^n) = 0$ ,  $n = 0, \pm 1, \pm 2, \dots$ . Then

$$0 = \Phi((f_1^*)^m) = \Phi(f_m^*) = \int_{-\infty}^{\infty} f_m(x) \phi(x) dx, m \geq 0$$

or

$$0 = \int_0^{\infty} x^{m-1} e^{-x} \phi(x) dx, m = 1, 2, \dots$$

Hence  $\phi(x) \equiv 0$ ,  $x > 0$ . Similarly  $\phi(x) \equiv 0$ ,  $x < 0$ . Thus  $\Phi \equiv 0$  and so  $\{\lambda^n \mid -\infty < n < \infty\}$  spans  $B$  as claimed. Let now  $B^+$  be the algebra of functions in  $B$  admitting analytic extensions to  $|\lambda| < 1$ . Since  $B$  satisfies (1), as we have just seen,  $B^+$  is maximal in  $B$  by the Lemma.

We now claim that  $B^+ = \{f^* + c \mid f \in L^+, c \text{ constant}\}$ . If  $f \in L$ , the function  $F$  defined by

$$F(s) = \int_{-\infty}^{\infty} f(x)e^{-sx} dx, \quad s = it, \quad t \text{ real}$$

is related to  $f^*$  by  $f^*(\lambda) = F((1+\lambda)/(1-\lambda))$ . Hence  $f^* + c \in B^+$  if and only if  $F$  admits an analytic extension to the right half-plane, continuous in the closed right half-plane, including  $\infty$ . It is clear that this occurs if and only if  $f \in L^+$ . Thus  $B^+ = \{f^* + c \mid f \in L^+, c \text{ constant}\}$ , as asserted. Let  $K$  be any closed proper subalgebra of  $L$  containing  $L^+$ . Put  $K^* = \{f^* + c \mid f \in K, c \text{ constant}\}$ . Then  $K^*$  is a closed proper subalgebra of  $B$ , and since  $K \supseteq L^+$ ,  $K^* \supseteq \{f^* + c \mid f \in L^+, c \text{ constant}\}$  which is  $B^+$ , as we just saw. Since  $B^+$  is maximal,  $K^* = B^+$ , and so  $K = L^+$ . This proves the theorem.

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# LIMIT PROPERTIES AT ZERO OF THE MARKOV SEMI-GROUP.\*

By RAFAEL V. CHACON.<sup>1</sup>

**Introduction.** Let  $(\Omega, \mathcal{F}, P)$  be a probability triple, and  $\mathcal{B}$  be a Borel field of sets of points of  $X$ , and  $\{x(t, \omega), t \in (0, +\infty)\}$  be a Markov process with state space  $X$ , and with (stationary) transition (probability) function  $P_t(x, A)$ . In other words, there exists a function  $P_t(x, A)$ , defined for  $t \in (0, +\infty)$ ,  $x \in X$ , and  $A \in \mathcal{B}$  such that

$$P\{x(t+s) \in A \mid x(s, \omega)\} = P_t(x(s, \omega), A), \text{ p. 1.}$$

We suppose further that the transition function satisfies:

- (0.1) For each  $t \in (0, +\infty)$ ,  $x \in X$ ,  $P_t(x, \cdot)$  is a probability measure on  $\mathcal{B}$ .
- (0.2) For each  $t \in (0, +\infty)$ ,  $A \in \mathcal{B}$ ,  $P_t(\cdot, A)$  is  $\mathcal{B}$ -measurable.
- (0.3) The Chapman-Kolmogorov equation

$$P_{t+s}(x, A) = \int_X P_t(y, A) P_s(x, dy).$$

An additional condition is often imposed as well:

$$(0.4') \quad \lim_{t \rightarrow 0} P_t(x, A) = 1 \quad \text{if } x \in A.$$

We do not impose hypothesis (0.4'); our goal is to investigate to what extent this condition is satisfied under some minimal additional assumptions. In an earlier paper [1] it has been shown that if a condition like (0.4') is satisfied, then the transition functions will be continuous in  $t$ . In view of this fact, we suppose that the transition functions satisfy also:

- (0.4) For each  $x \in X$  and  $A \in \mathcal{B}$ ,  $P_t(x, A)$  is a continuous function of  $t$ .

If we suppose that  $X$  is the set of positive integers, and that  $\mathcal{B}$  is the Borel field of subsets of  $X$ , then the process is called a Markov chain. Doob [3] has proved the following theorem:

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THEOREM 0.1. *A Markov chain has continuous transition functions if and only if the integers may be divided into pairwise disjoint sets  $F, I_1, I_2, \dots$ , such that*

- (i)  $P_t(i, \{j\}) = 0, j \in F,$
- (ii)  $\lim_{t \rightarrow 0} P_t(i, I_j) = \delta_{I_j}(i), i \in \bigcup_{j=1}^{\infty} I_j,$
- (iii)  $P_t(i_1, \{j\}) = u_j P_t(i_2, I_k), j \in I_k, i_1, i_2 \in I_l,$
- (iv)  $P_t(i, \{j\}) = u_j P_t(i, I_k), j \in I_k, i \in F,$

where  $\{u_j\}$  is a sequence of non-negative numbers, and

- (v)  $\lim_{t \rightarrow 0} P_t(i, \{j\})$  exists,  $i, j = 1, 2, \dots$ .

We define the function  $\delta$  as follows:

$$\delta_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

We establish a result analogous to Theorem 0.1, assuming that the transition probabilities satisfy conditions (0.1), (0.2), (0.3) and (0.4). In the special case considered in Theorem 0.1, the function  $U$  which we obtain can be further analyzed, and the details of Theorem 0.1 can thereby be obtained. We remark that the method of proof is different from Doob's and that his proof doesn't seem to generalize.

It is possible to define certain semi-groups, using the transition functions, on suitably chosen Banach spaces. However, Hille's work on the existence of the identity as a limit as the parameter tends to zero does not yield our result in any obvious way.

**1. Results and proofs.** We suppose in what follows that we have a Markov process whose transition function satisfies (0.1), (0.2), (0.3), and (0.4). Blackwell [2] gave a special case of Lemma 1.1 to study idempotent Markov chains.

LEMMA 1.1. *If  $A$  and  $B$  are two sets in  $\mathcal{B}$ , then (where  $\psi_C(\cdot)$  stands for the characteristic function of the set  $C$ )*

$$\begin{aligned} & \int_B \left\{ \int_{cB} P_s(z, A) P_t(y, dz) \right\} P_u(x, dy) \\ &= \int \psi_{cB}(z) P_s(z, A) P_{t+u}(x, dz) - \int_{cB} \left\{ \int_{cB} P_s(z, A) P_t(y, dz) \right\} P_u(x, dy), \end{aligned}$$

and

$$\begin{aligned} & \int_{cB} \left\{ \int_B P_s(z, A) P_t(y, dz) \right\} P_u(x, dy) \\ &= \int \psi_{cB}(y) P_{s+t}(y, A) P_u(x, dy) - \int_{cB} \left\{ \int_{cB} P_s(z, A) P_t(y, dz) \right\} P_u(x, dy). \end{aligned}$$

*Proof.* Follows at once by transposing the second terms on the right to the left and combining each of the terms into single integrals. That the interchange of order of integration is valid can be easily shown.

LEMMA 1.2. If  $A$  and  $B$  are two sets in  $\mathfrak{B}$ , and if  $a$  is a real number, and  $s > 0$ ,  $t > 0$ ,  $u > 0$ ,

$$\begin{aligned} & \lim_{t \rightarrow 0} \left\{ \int_B \left[ \int_{cB} (P_s(z, A) P_t(y, dz)) \right] P_u(x, dy) \right. \\ & \quad \left. - \int_{cB} \left[ \int_B (P_s(z, A) - a) P_t(y, dz) \right] P_u(x, dy) \right\} = 0. \end{aligned}$$

*Proof.* Follows at once from Lemma 1.1.

LEMMA 1.3. For each  $s > 0$ ,  $u > 0$ ,  $c$  real,  $x \in X$  and  $A \in \mathfrak{B}$ , fixed, if  $C = \{z: P_s(z, A) < c\}$  then

$$\lim_{t \rightarrow 0} P_t(y, C) = \delta_C(y) \text{ in } P_u(x, \cdot)\text{-measure.}$$

The lemma remains valid if " $<$ " is replaced by either " $\leq$ " or " $>$ " or " $\geq$ " in the definition of the set  $C$ .

*Proof.* Using Lemma 1.2, put  $B = B_n = \{z: P_s(z, A) > c - 1/n\}$  and  $a = a_n = c - 1/n$ . Both terms have the same sign in this application, and hence each must have a zero limit. We have, using the second term,

$$\lim_{t \downarrow 0} \int \psi_{cB_n}(y) \left[ \int \psi_{B_n}(z) \{P_s(z, A) - c + 1/n\} P_t(y, dz) \right] P_u(x, dy) = 0.$$

Since the inner integral is of constant sign as a function of  $y$ , we have that

$$\psi_{cB_n}(y) \int \psi_{B_n}(z) \{P_s(z, A) - c + 1/n\} P_t(y, dz)$$

converges in  $P_u(x, \cdot)$ -measure to zero as  $t \rightarrow 0$ . From this it follows that

$$\psi_{cB_n}(y) P_t(y, \{z: P_s(z, A) \geq c\})$$

also converges in  $P_u(x, \cdot)$ -measure to zero as  $t \rightarrow 0$ , and from this that

$$(1.3.1) \quad \psi_{\{y: P_s(y, A) < c\}}(y) P_t(y, \{z: P_s(z, A) \geq c\})$$

converges in  $P_u(x, \cdot)$ -measure to zero as  $t \rightarrow 0$  as well. This clearly implies that

$$(1.3.2) \quad \psi_{\{y: P_s(y, A) < c\}}(y) P_t(y, \{z: P_s(z, A) < c\})$$

converges in  $P_u(x, \cdot)$ -measure to one as  $t \rightarrow 0$ . Now using the first part of Lemma 1.1 with  $A = X$ ,  $B = \{y: P_s(y, A) \geq c\}$ , we have that

$$\begin{aligned} & \int_B P_t(y, \{z: P_s(z, A) < c\}) P_u(x, dy) \\ &= P_{t+u}(x, \{z: P_s(z, A) < c\}) - \int_{cB} P_t(y, \{z: P_s(z, A) < c\}) P_u(x, dy), \end{aligned}$$

from which it follows that

$$\psi_{\{y: P_s(y, A) \geq c\}}(y) P_t(y, \{z: P_s(z, A) < c\})$$

tends to zero in  $P_u(x, \cdot)$ -measure as  $t \rightarrow 0$ . The " $>$ " part of the lemma follows by a similar argument, and the " $\leq$ " and " $\geq$ " parts from these by taking complements.

LEMMA 1.4. If  $\{A_n\}$  is a sequence of sets such that for each  $n$  and for some fixed  $u > 0$  and  $x \in X$ ,  $P_t(y, A_n)$  converges to  $\delta_{A_n}(y)$  in  $P_u(x, \cdot)$ -measure as  $t \rightarrow 0$ , then it follows that  $P_t(y, B)$  converges to  $\delta_B(y)$  in  $P_u(x, \cdot)$ -measure as well, where  $B = \bigcup_{n=1}^{\infty} A_n$ .

Proof. Note first that, letting  $B_k = \bigcup_{n=1}^k A_n$ ,

$$\begin{aligned} \int_{B_k} \{1 - P_t(y, B_k)\} P_u(x, dy) &\leq \sum_{n=1}^k \int_{A_n} \{1 - P_t(y, B_k)\} P_u(x, dy) \\ &\leq \sum_{n=1}^k \int_{A_n} \{1 - P_t(y, A_n)\} P_u(x, dy), \end{aligned}$$

and that

$$\begin{aligned} \int_{cB_k} P_t(y, B_k) P_u(x, dy) &= \sum_{n=1}^k \int_{B_k} P_t(y, A_n) P_u(x, dy) \\ &\leq \sum_{n=1}^k \int_{cA_n} P_t(y, A_n) P_u(x, dy), \end{aligned}$$

and thus that the result of the lemma follows for finite sums. Next, to see the result for countably infinite sums, note that for each  $k > 1$ ,

$$\int_B \{1 - P_t(y, B)\} P_u(x, dy) \leq \int_B \{1 - P_t(y, B_k)\} P_u(x, dy).$$

Hence, using the result for finite sums, we have for each  $k > 1$ ,

$$\int_B \{1 - P_t(y, B)\} P_u(x, dy) \leq P_u(x, B_k),$$

and thus that

$$(1.4.1) \quad \lim_{t \rightarrow 0} \int_B \{1 - P_t(y, B)\} P_u(x, dy) = 0.$$

It remains to show that

$$\lim_{t \rightarrow 0} \int_{cB} P_t(y, B) P_u(x, dy) = 0.$$

From the first part of Lemma 1.1, putting  $A = X$ , and interchanging  $B$  with  $cB$ , we have that

$$\int_{cB} P_t(y, B) P_u(x, dy) = P_{t+u}(x, B) - \int_B P_t(y, B) p_u(x, dy)$$

and the desired result follows from this, and (1.4.1).

**Definition 1.1.** Let  $\mathcal{G}$  be the Borel field generated by sets of the form  $\{x: P_t(x, A) \in (a, b]\}$  for each  $t > 0$ ,  $A \in \mathcal{B}$  and  $a$  and  $b$  real.

**THEOREM 1.1.** If  $G \in \mathcal{G}$ ,  $x \in X$ , and  $u > 0$ , then  $P_t(y, G)$  converges to  $\delta_G(y)$  in  $P_u(x, \cdot)$ -measure, as  $t \rightarrow 0$ .

*Proof.* Since if  $P_t(y, A)$  converges to  $\delta_A(y)$  in  $P_u(x, \cdot)$ -measure as  $t \rightarrow 0$  then  $P_t(y, cA)$  converges to  $\delta_{cA}(y)$  in  $P_u(x, \cdot)$ -measure as  $t \rightarrow 0$  as well, it follows easily from Lemma 1.4 that the class of sets for which the assertion holds is a monotone class. That it includes finite unions of sets of the form  $\{x: P_t(x, A) \in (a, b]\}$  follows from Lemmas 1.3 and 1.4.

**LEMMA 1.5.** If  $\{(X, \mathcal{B}, \mu_\gamma), \gamma \in \Gamma\}$  is a family of bounded measure spaces, and if  $\{f_n(x)\}$  is a sequence of  $\mathcal{B}$ -measurable function which, for each  $\gamma \in \Gamma$ , converges in  $\mu_\gamma$ -measure, then there exists a function  $f(x)$ , measurable with respect to the Borel field  $\bigcap_{\gamma \in \Gamma} \mathcal{B}(\mu_\gamma)$ , where  $\mathcal{B}(\mu_\gamma)$  denotes the completion of  $\mathcal{B}$  with respect to  $\mu_\gamma$ , such that for each  $\gamma \in \Gamma$ ,  $\{f_n(x)\}$  converges in  $\mu_\gamma$ -measure to  $f(x)$ .

*Proof.* For each fixed  $\gamma \in \Gamma$  there exists a subsequence  $\{n^\gamma(k)\}$  and a set  $A_\gamma$  such that

- (i)  $A_\gamma = \{x: \lim_{k \rightarrow \infty} f_{n^\gamma(k)}(x) \text{ exists}\},$
- (ii)  $\mu_\gamma(A_\gamma) = 1,$
- (iii)  $f_n(x)$  converges in  $\mu_\gamma$ -measure to  $\lim_{k \rightarrow \infty} f_{n^\gamma(k)}(x)$  on  $A_\gamma.$

Clearly, the family  $\{A_\gamma, \gamma \in \Gamma\}$  of sets of  $\mathcal{B}$  has at most continuum distinct elements, since there are at most continuum distinct subsequences  $\{n^\gamma(k)\}$ ,  $\gamma \in \Gamma$ . For this reason there exists a well-ordering of these sets such that for each  $A_{\gamma'}$  there are at most countably many distinct sets  $A_\gamma, A_\gamma < A_{\gamma'}$ . On

$$A'_{\gamma'} = A_{\gamma'} - \bigcup_{A_\gamma < A_{\gamma'}} A_\gamma,$$

define  $f(x)$  by putting  $f(x) = \lim_{k \rightarrow \infty} f_{n^{\gamma'}(k)}(x)$ . On  $X - \bigcup_{\gamma \in \Gamma} A_\gamma$  define  $f(x)$  by putting  $f(x) = 0$ . It follows that  $f(x)$  is defined everywhere, and by (ii) it follows that  $f(x)$  is measurable with respect to  $\bigcap_{\gamma \in \Gamma} \mathcal{B}(\mu_\gamma)$ .

To see that for each  $\gamma' \in \Gamma$ ,  $\{f_n(x)\}$  converges in  $\mu_{\gamma'}$ -measure to  $f(x)$ , note that for each  $\epsilon > 0$ ,

$$\begin{aligned} \lim_{n \rightarrow \infty} \mu_{\gamma'}\{|f_n - f| > \epsilon\} &= \lim_{n \rightarrow \infty} \mu_{\gamma'}\{A_{\gamma'} \cap \{x: |f_n - f| > \epsilon\}\} \\ &= \lim_{n \rightarrow \infty} \mu_{\gamma'}\left\{\bigcup_{A_\gamma \leq A_{\gamma'}} A'_\gamma \cap \{x: |f_n - f| > \epsilon\}\right\} \\ &= \lim_{n \rightarrow \infty} \sum_{A'_\gamma \leq A_{\gamma'}} \mu_{\gamma'}\{A'_\gamma \cap \{x: |f_n - f| > \epsilon\}\}. \end{aligned}$$

Note that there are at most countably many terms in the last sum, and that each term of the sum tends to zero as  $n \rightarrow \infty$ . That the whole sum tends to zero follows from the fact that

$$\sum_{A'_\gamma \leq A_{\gamma'}} \mu_{\gamma'}\{A'_\gamma\} = K,$$

when  $K$  is the bound on the measures.

**Definition.** Let  $\mathcal{G}_1 = \bigcap_{\substack{x \in X \\ t \in (0, +\infty)}} \mathcal{G}_{(x,t)}$ , where  $\mathcal{G}_{(x,t)}$  denotes the completion of  $\mathcal{G}$  with respect to  $P_t(x, \cdot)$ , and let  $\mathcal{B}_1$  be the smallest Borel field containing  $\mathcal{G}_1$  and  $\mathcal{B}$ .

**THEOREM 1.2.** If for each  $A \in \mathcal{B}$ ,  $u > 0$ , and  $x \in X$  we have that  $P_t(y, A)$  converges, as  $t \rightarrow 0$ , in  $P_u(x, \cdot)$  measure, then there exist functions  $U(y, A)$ , defined for all  $y \in X$  and  $A \in \mathcal{B}_1$  such that:

- (i)  $U(\cdot, A)$  is  $\mathcal{G}_1$ -measurable, for each  $A \in \mathcal{B}$ ,
- (ii)  $U(y, G) = \delta_G(y)$ , for each  $G \in \mathcal{G}_1$ ,
- (iii)  $U(y, \bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} U(y, A_i)$

for every sequence  $\{A_i\}$  of pairwise disjoint sets of  $\mathcal{B}_1$  except on a set  $N$

(depending on  $\{A_i\}$ ) such that  $P_t(x, N) = 0$  for each  $t > 0$  and  $x \in X$ , and

$$(iv) \lim_{s \rightarrow 0} \int |P_s(y, A) - U(y, A)| P_t(x, dy) = 0,$$

for each  $x \in X$ ,  $t > 0$ , and  $A \in \mathcal{B}$ .

*Proof.* Define  $U(x, G) = \delta_G(x)$  for each  $G \in \mathcal{G}_1$ , and choose a version of the limit measure for  $U(x, A)$ , for  $A \in \mathcal{B}_1$  and  $A \notin \mathcal{G}_1$ , as is possible by Lemma 1.5. (i), (ii), (iii), and (iv) follow by Lemma 1.5, definition, by a neasy proof, and hypothesis, respectively.

**COROLLARY 1.1.** *Under the hypothesis of Theorem 1.2, if  $A \in \mathcal{B}$  and  $\epsilon > 0$ , then there exist positive constants  $a_1, a_2, \dots, a_N$ , and sets  $A_1, \dots, A_N$  of  $\mathcal{G}$  such that*

$$|P_t(x, A) - \sum_{k=1}^N a_k P_t(x, A_k)| < \epsilon$$

*uniformly in  $x$  and  $t$ .*

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## COLLINEATION GROUPS OF NON-DESARGUESIAN PLANES II.\*

### Some seminuclear division algebras.

By D. R. HUGHES.<sup>1</sup>

**1. Introduction.** We will consider a class of non-associative division algebras and the projective planes they coordinatize, with the aim of determining the collineation groups of the planes. This will be the second class of finite division ring planes so analyzed, and we will see that like the "twisted fields" of Albert ([2]), the groups are solvable. This is in sharp contrast to the finite Hall Veblen-Wedderburn planes and the (non-Veblen-Wedderburn) Hughes planes, both of which have non-solvable collineation groups ([4, 6]). Since both the Hall and Hughes planes appear more removed from the Desarguesian case than any division ring plane, it is striking that all the known finite non-associative division ring planes have solvable groups.

Our treatment of the collineation group leaves unanswered a number of possibly interesting questions: (1) Since we only determine a sub-normal series for the group, is the group itself amenable to direct computation? (2) What is the transitive structure of the group, and more particularly, what are the transitive constituents on the line at infinity of the autotopism group  $\mathfrak{G}$ ?

**2. Preliminary discussion.** Let  $R$  be a division ring; i.e.,  $(R, +)$  is an abelian group, both distributive laws hold, there is a multiplicative identity  $1 (\neq 0)$ , and every equation  $ax = b$  ( $ya = b$ ), for  $a \neq 0$ , has a unique solution for  $x$  (for  $y$ ). Let  $T$  be an additive one-to-one mapping of  $R$  upon  $R$ , and  $a, b$  non-zero elements of  $R$ ; the triple  $(T, a, b)$  is called an *autotopism* of  $R$  if

$$(1) \quad (xy)T = (xa)T(by)T, \text{ for all } x, y \text{ in } R.$$

Let us suppose that  $R$  is non-alternative, and let  $\pi$  be the projective

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plane coordinatized by  $R$  (see [4, 5] for construction of  $\pi$  from  $R$ ). Then there is a metabelian group  $\mathfrak{G}_0$ , generated by "translations" and "shears," such that if  $\mathfrak{G}_1$  is the full collineation group of  $\pi$ , then  $\mathfrak{G}_0$  is normal in  $\mathfrak{G}_1$  and  $\mathfrak{G}_1/\mathfrak{G}_0$  is isomorphic to the group  $\mathfrak{G}$  of all autotopisms of  $R$  (see [2, 5]). So we will content ourselves with studying  $\mathfrak{G}$ . We remark that in case  $R$  is finite, to say  $R$  is non-alternative is equivalent to saying that it is not a field.

Now we introduce some notation. The *right nucleus*  $N_r$  of  $R$  is the set of all  $n$  in  $R$  such that  $(xy)n = x(yn)$ , for all  $x, y$  in  $R$ ; the *middle* and *left* nuclei are defined analogously. The *nucleus* of  $R$  is the intersection of the three one-sided nuclei, while the *center* of  $R$  is the set of all  $z$  in the nucleus such that  $xz = zx$  for all  $x$  in  $R$ . It is well-known that all five of these subsets are themselves division rings (even associative, of course).

If  $x$  is in  $R$ , define the mappings  $R_x$  and  $L_x$  of  $R$  by  $yR_x = yx$ ,  $yL_x = xy$ . If  $x \neq 0$ , these mappings are non-singular and have inverses. Sometimes we will write  $R(x)$  for  $R_x$ , etc.

Now let  $(T, a, b)$  be an autotopism of  $R$ , and for each  $n$  in  $N_r$ , define  $n^\alpha = (ban)T$ . Then (see [3, p. 250]),  $\alpha$  is an automorphism of  $N_r$  onto  $N_r$ ; let  $x$  be in  $R$ ,  $n$  in  $N_r$ . Then:

$$(xn)T = [(xR_a^{-1} \cdot a)n]T = [(xR_a^{-1})(an)]T,$$

and so, from (1):

$$(xn)T = [(xR_a^{-1})a]T(ban)T = (xT)n^\alpha.$$

**LEMMA 1.** *Every autotopism of  $R$  is a semi-linear transformation over the right nucleus.<sup>2</sup>*

Now we define our class of division rings. Let  $F$  be a field,  $\sigma$  a non-identity automorphism of  $F$ , and  $\delta_0$  and  $\delta_1$  elements of  $F$  such that

$$(2) \quad \delta_0 \neq w^{1+\sigma} + \delta_1 w, \text{ for any } w \text{ in } F.$$

Note that (2) requires  $\delta_0 \neq 0$ . Furthermore, if  $\sigma^2 = 1$ ,  $\delta_1 = 0$ , and  $F$  is finite, then  $\delta_0 \neq \delta_0^\sigma$ . For the elements  $x^{1+\sigma}$ , as  $x$  ranges over  $F$ , range over the subfield  $K$  of all elements fixed by  $\sigma$ , and so  $\delta_0$  cannot be in this subfield. Let  $R$  be a two-dimensional vector space over  $F$ , with basis elements  $1, \lambda$ . Define multiplication in  $R$  by

$$(3) \quad (x + \lambda y)(u + \lambda v) = (xu + \delta_0 y^\sigma v) + \lambda(yu + x^\sigma v + \delta_1 y^\sigma v).$$

This corresponds to demanding that  $\lambda^2 = \delta_0 + \lambda\delta_1$ ,  $x\lambda = \lambda x^\sigma$ , and that  $F$  be

<sup>2</sup> There is an analogous theorem for the left nucleus, and even for the center.

the right and middle nuclei of  $R$ . Then  $R$  is a division ring<sup>3</sup>; for the rest of the paper, we shall assume that  $R$  is finite.

We wish to consider the space  $\mathcal{S}$  of all additive operators on  $F$ , and in particular that subspace of  $\mathcal{S}$  containing all the mappings  $R_x$ ,  $x$  in  $F$ , as well as the mappings  $Q: x \rightarrow x^\mu$ , where  $\mu$  is an automorphism of  $F$ . We will further consider the set of 2 by 2 matrices over  $\mathcal{S}$ , considered as acting on elements of  $R$ , where  $x + \lambda y$  is identified with the vector  $(x, y)$ .

Let  $\sigma$  be the defining automorphism of  $R$ , and  $S$  the element of  $\mathcal{S}$  defined by  $S: x \rightarrow x^\sigma$ ; for an autotopism  $(T, a, b)$  of  $R$ , let  $\alpha$  be the automorphism of the right nucleus of  $R$ , as in Lemma 1 and the preceding discussion. Since  $F = N_r$ ,  $\alpha$  is an automorphism of  $F$ ; define  $A: x \rightarrow x^\alpha$ . For simplicity, write  $\gamma = \alpha^{-1}$ .

Lemma 1 asserts that  $T$  has the form  $AT_1$ , where

$$T_1 = \begin{bmatrix} R(a_0) & R(a_1) \\ R(b_0) & R(b_1) \end{bmatrix}.$$

Furthermore, from (3) we can write:

$$R_{u+\lambda v} = R(u)I + SR(v)E,$$

where the terms on the right are understood to be among the two-dimensional matrices over  $\mathcal{S}$ , and the term on the left is the ordinary right multiplication in  $R$ . Here,  $E$  and  $I$  are defined by:

$$I = \begin{bmatrix} R(1) & 0 \\ 0 & R(1) \end{bmatrix}, \quad E = \begin{bmatrix} 0 & R(1) \\ R(\delta_0) & R(\delta_1) \end{bmatrix}.$$

Now we note that:

$$(4) \quad AR(x) = R(x^\gamma)A$$

$$(5) \quad R(x)S = SR(x^\sigma).$$

Furthermore,  $S$  and  $A$  commute with each other, since  $F$  is a Galois field.

Let the  $a$  of  $(T, a, b)$  be  $a = r + \lambda s$ , and write  $yP = (by)T$ ; then (1) is equivalent to:

$$(6) \quad R_{r+\lambda s}TR_{yP} = R_yT.$$

Let  $Q = P^{-1}$ ; then (6) becomes:

$$(7) \quad R_{r+\lambda s}TR_{u+\lambda v}T^{-1} = R_{(u+\lambda v)Q},$$

<sup>3</sup> In a forthcoming paper, these division rings are studied in more detail by Edwin Kleinfeld and the present writer.

or:

$$(8) \quad [R(r)I + SR(s)E]AT_1[R(u)I + SR(v)E]T_1^{-1}A^{-1} = R_{(u+\lambda v)}Q.$$

**3. Computation of the autotopism group.** If we multiply out the left side of (8) and use (4) and (5), we find:

$$(9) \quad R(ru^\gamma)I + S[R(r^\sigma v^\gamma)T_1^{\sigma\gamma}E^\gamma T_1^{-\gamma} + R(su^\gamma)E] + S^2R(s^\sigma v^\gamma)E^\sigma T_1^{\sigma\gamma}E^\gamma T_1^{-\gamma} \\ = R_{(u+\lambda v)}Q.$$

Here  $T_1^\gamma$ , for instance, means:

$$T_1^\gamma = \begin{bmatrix} R(a_0^\gamma) & R(a_1^\gamma) \\ R(b_0^\gamma) & R(b_1^\gamma) \end{bmatrix}.$$

Since the left side of (9) must be a right multiplication, it must have the form  $R(x)I + SR(y)E$ , for some  $x, y$  in  $F$ . Now we must distinguish cases.

If  $S^2 \neq 1$  (i.e., if  $\sigma^2 \neq 1$ ), then the last term on the left of (9) can only be identically zero; this implies  $s = 0$ . Then (9) becomes:

$$(10) \quad R(ru^\gamma)I + SR(r^\sigma v^\gamma)T_1^{\sigma\gamma}E^\gamma T_1^{-\gamma} = R_{(u+\lambda v)}Q.$$

So the second term in (10) has the form  $SR(y)E$ , and hence:

$$(11) \quad T_1^{\sigma\gamma}E^\gamma = R(y)ET_1^\gamma, \text{ for some } y \neq 0.$$

On the other hand, if  $\sigma^2 = 1$ , then  $S^2 = 1$ , so (9) becomes:

$$(12) \quad R(ru^\gamma)I + R(s^\sigma v^\gamma)E^\sigma T_1^{\sigma\gamma}T_1^{-\gamma} + S[R(r^\sigma v^\gamma)T_1^{\sigma\gamma}E^\gamma T_1^{-\gamma} + R(su^\gamma)E] \\ = R_{(u+\lambda v)}Q.$$

In the following, we can simplify matters by understanding that:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 \\ \delta_0 & \delta_1 \end{bmatrix}, \quad T_1 = \begin{bmatrix} a_0 & a_1 \\ b_0 & b_1 \end{bmatrix}.$$

Then (12) implies:

$$(13) \quad ru^\gamma I + s^\sigma v^\gamma E^\sigma T_1^{\sigma\gamma}E^\gamma T_1^{-\gamma} = kI,$$

$$(14) \quad r^\sigma v^\gamma T_1^{\sigma\gamma}E^\gamma T_1^{-\gamma} + su^\gamma E = mE,$$

for some choice of  $k, m$  in  $F$ . So we write, from (13) and (14):

$$(15) \quad s^\sigma v^\gamma E^\sigma T_1^{\sigma\gamma}E^\gamma T_1^{-\gamma} = (k - ru^\gamma)I,$$

$$(16) \quad r^\sigma v^\gamma T_1^{\sigma\gamma}E^\gamma T_1^{-\gamma} = (m - su^\gamma)E.$$

Then (15) and (16) imply:

$$(17) \quad s^\sigma(m - su^\gamma)E^\sigma E = r^\sigma(k - ru^\gamma)I.$$

Now note that

$$E^\sigma E = \begin{bmatrix} \delta_0 & \delta \\ \delta_0 \delta_1^\sigma & \delta_0^\sigma + \delta_1^{1+\sigma} \end{bmatrix}.$$

So the left side of (17) is not scalar unless it is 0 or  $\delta_1 = 0$  and  $\delta_0 = \delta_0^\sigma$ ; as remarked earlier, this latter possibility violates (2) since  $\sigma^2 = 1$ . But the right side of (17) is scalar, so either one of  $r, s$  is zero, or  $m = su^\gamma$  and  $k = ru^\gamma$ . Suppose the latter of these two possibilities occurs; then  $(u + \lambda v)Q = k + \lambda m = (r + \lambda s)u^\gamma$ , and this implies that  $Q$  is singular, which is impossible. Hence:

LEMMA 2. *If  $\sigma^2 = 1$ , then  $r = 0$  or  $s = 0$ .*

If  $s = 0$  then it is easily seen that we are led to equation (11), and so no distinction is necessary between  $\sigma^2 = 1$  and  $\sigma^2 \neq 1$ . On the other hand, if  $r = 0$ , the additional equation necessary is:

$$(18) \quad E^\sigma T_1^\sigma E^\gamma = n T_1^\gamma, \text{ for some } n \neq 0.$$

Now suppose we have two autotopisms  $A_1 T_1$  and  $A_2 T_2$  (strictly, we are looking only at the additive map part of an autotopism triple). Then one easily computes that  $(A_1 T_1)(A_2 T_2) = (A_1 A_2)(T_1^\alpha T_2)$ , where  $\alpha$  is the automorphism of  $N_r$  associated with the autotopism  $A_2 T_2$ . Remembering that the  $\gamma$  in (18) is actually  $\alpha^{-1}$ , it is straightforward to show:

LEMMA 3. *The product of two autotopisms satisfying (18) is an autotopism with  $s = 0$  and satisfying (11).*

So if we disregard the autotopisms satisfying (18), we will be considering a subgroup  $\bar{\mathcal{G}}$  of  $\mathcal{G}$ , where  $\mathcal{G}/\bar{\mathcal{G}}$  has order one or two. We shall restrict attention to this group  $\bar{\mathcal{G}}$ , and thus to autotopisms satisfying:

$$(19) \quad T^\sigma \gamma E^\gamma = k E T^\gamma,$$

where  $k$  is in  $F$ ,  $k \neq 0$ , and  $E$  and  $T$  are the matrices  $E$  and  $T_1$  following equation (12).

Among all solutions of (19), the set  $\mathfrak{S}_1$  of those  $T$  for which  $\gamma = 1$  is a normal subgroup, whose factor group is contained in the group of all semi-linear transformations over  $F$  modulo the group of linear transformation over  $F$ . I. e.,  $\bar{\mathcal{G}}/\mathfrak{S}_1$  is isomorphic to a subgroup of the automorphism group of  $F$ ; since  $F$  is a Galois field,  $\bar{\mathcal{G}}/\mathfrak{S}_1$  is cyclic.

So  $\mathfrak{S}_1$  consists of the  $T$  satisfying:

$$(20) \quad T^\sigma E = kET.$$

But among all elements of  $\mathfrak{S}_1$ , the subset  $\mathfrak{S}$  of all solutions of (20) for which  $k = 1$  is a normal subgroup (of  $\mathfrak{S}_1$ ), and  $\mathfrak{S}_1/\mathfrak{S}$  is isomorphic to a subgroup of the multiplicative group of  $F$ . (This can be easily seen by considering the mapping which sends a solution of (20) onto the element  $k$  associated with it; the mapping is a homomorphism.)

Now we have reduced our problem to the point where some direct computing is possible.  $\mathfrak{S}$  consists of all (non-singular) 2 by 2 matrices  $T$  over  $F$  which satisfy:

$$(21) \quad T^\sigma E = ET.$$

Then (21) immediately implies:

$$(22) \quad b_0 = a_1^\sigma \delta_0,$$

$$(23) \quad b_1 = a_0^\sigma + a_1^\sigma \delta_1,$$

$$(24) \quad a_0^{\sigma^2} + a_1^{\sigma^2} \delta_1^\sigma = a_0 + a_1^\sigma \delta_1,$$

$$(25) \quad a_1^{\sigma^2} \delta_0^\sigma + a_0^{\sigma^2} \delta_1 + a_1^{\sigma^2} \delta_1^{1+\sigma} = a_1 \delta_0 + a_0^\sigma \delta_1 + a_1^\sigma \delta_1^2$$

So if  $T$  is in  $\mathfrak{S}$ , we can represent  $T$  by the vector  $(a_0, a_1)$ , consisting of its first row, where  $a_0, a_1$  satisfy (24) and (25). The multiplication of these vectors (i.e., the multiplication of the elements of  $\mathfrak{S}$ ) is:

$$(26) \quad (a_0, a_1)(c_0, c_1) = (a_0 c_0 + \delta_0 a_1 c_1^\sigma, a_0 c_1 + a_1 c_0^\sigma + \delta_1 a_1 c_1^\sigma),$$

as one sees by multiplying together two matrices satisfying (21) and using (22) and (23).

Now consider the division ring  $R'$  which is anti-isomorphic to  $R$ ; we can represent the multiplication in  $R'$  by:

$$(x + y\lambda)(u + v\lambda) = (xu + \delta_0 yv^\sigma) + (xv + yu^\sigma + \delta_1 yv^\sigma)\lambda.$$

But this agrees with (26) if we identify  $(a_0, a_1)$  with  $a_0 + a_1\lambda$ , and so  $\mathfrak{S}$  is isomorphic to a subgroup of the multiplicative loop of  $R'$ . In order to identify  $\mathfrak{S}$  more precisely, we find it convenient to compute the right nucleus of  $R'$ .

For any three elements  $a, b, c$  of  $R'$ , let  $(a, b, c) = (ab)c - a(bc)$ . Then it is straightforward computation that:

$$\begin{aligned} (x + y\lambda, u + v\lambda, w + z\lambda) &= yv^\sigma \delta_0 [w - w^{\sigma^2} + \delta_1 z^\sigma - \delta_1 z^{\sigma^2}] \\ &\quad + yv^\sigma [\delta_0 z - \delta_0^\sigma z^{\sigma^2} + \delta_1 (\delta_1 z^\sigma - \delta_1^\sigma z^{\sigma^2}) + \delta_1 (w^\sigma - w^{\sigma^2})] \lambda. \end{aligned}$$

So  $w + z\lambda$  is in the right nucleus of  $R'$  if and only if:

$$(27) \quad w + \delta_1 z^\sigma = w^{\sigma^2} + \delta_1^\sigma z^{\sigma^2},$$

$$(28) \quad \delta_0 z + \delta_1^2 z^\sigma + \delta_1 w^\sigma = \delta_0^\sigma z^{\sigma^2} + \delta_1^{1+\sigma} z^{\sigma^2} + \delta_1 w^{\sigma^2}.$$

But (27), (28) become equations (24), (25) with the change  $a_0 = w$ ,  $a_1 = z$ , and thus  $\mathfrak{S}$  is isomorphic to the multiplicative group of the right nucleus of  $R'$ . Anti-isomorphic groups are isomorphic, so this means that  $\mathfrak{S}$  is isomorphic to the multiplicative group of the left nucleus of  $R$ ; the left nucleus is finite and associative, hence is a Galois field, so it has a cyclic multiplicative group.

**THEOREM 1.** *The autotopism group  $\mathfrak{G}$  of  $R$  is solvable when  $R$  is finite, and  $\mathfrak{G}$  has a sub-normal series:*

$$\mathfrak{G} \supseteq \bar{\mathfrak{G}} \supseteq \mathfrak{S}_1 \supseteq \mathfrak{S} \supseteq 1,$$

where (i)  $\mathfrak{G}/\bar{\mathfrak{G}}$  has order one or two, and has order one if  $\sigma^2 \neq 1$ ; (ii)  $\bar{\mathfrak{G}}/\mathfrak{S}_1$  is isomorphic to a subgroup of the automorphism group of  $F$ ; (iii)  $\mathfrak{S}_1/\mathfrak{S}$  is isomorphic to a subgroup of the multiplicative group of  $F$ ; (iv)  $\mathfrak{S}$  is isomorphic to the multiplicative group of the left nucleus of  $R$ .

**COROLLARY.** *If  $\pi$  is the projective plane coordinatized by the finite semi-nuclear division algebra  $R$ , then the collineation group of  $\pi$  is solvable.*

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# TAME COVERINGS AND FUNDAMENTAL GROUPS OF ALGEBRAIC VARIETIES.\*

## Part II: Branch Curves with Higher Singularities.

By SHREERAM ABHYANKAR.<sup>1</sup>

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**Introduction.** In this paper we study tame coverings and fundamental groups of a non-singular algebraic surface  $V$  (over an algebraically closed ground field  $k$ ) minus a curve  $W$  having arbitrary singularities; it is a continuation of Part I of this series [*American Journal of Mathematics*, vol. 81, 1959] in which we considered the situation when  $W$  had only strong normal crossings (there the dimension of  $V$  was arbitrary and  $W$  was of co-dimension one). Here, in the introduction, we shall briefly and approximately describe the main results and the contents of the various sections, referring for the precise definitions and statements to the body of this paper and to Part I. Denote by  $\pi'(V - W)$  the group tower of the galois groups over  $k(V)$  of all the finite galois extensions of  $k(V)$  (in some fixed algebraic closure of  $k(V)$ ), which are tamely ramified over  $V$  and for which the branch locus over  $V$  is contained in  $W$ . For an irreducible component  $W_1$  of  $W$  define the strength of singularities  $\nu(W_1, W; V)$  by  $\nu(W_1, W; V) = \frac{1}{2} \sum \mu_i(\mu_i + 1)$ , where the  $\mu_i$  are the multiplicities of the various points and "infinitely near" points of  $W_1$  at which  $W$  does not have a strong normal crossing. Let  $W_1, \dots, W_t$  be the irreducible components of  $W$ . Then the main results on fundamental groups in this paper are: (1) If  $V$  is simply connected and for some labelling of the components  $W_j$  we have  $\dim |W_j| > 1 + \nu(W_j, W_j \cup W_{j+1} \cup \dots \cup W_t; V)$  for  $j = 1, \dots, t$ , then  $\pi'(V - W)$  is generated by  $t$  generators, is  $t$ -step solvable, and has a weak parent group generated by  $t$  generators. (2) If  $V$  is simply connected and  $\dim |W_j| > 1 + \nu(W_j, W; V)$  for  $j = 1, \dots, t$ , then  $\pi'(V - W)$  is generated by  $t$  generators, is  $t$ -step nilpotent, and has a weak parent group generated by  $t$  generators; if in addition  $W_j$  and  $W_k$  have a point in common at which  $W$  has a normal crossing whenever  $j \neq k$ , then  $\pi'(V - W)$  is abelian.

An analysis of singularities only for curves on an algebraic surface would have been quicker, and would have been adequate for the above results on fundamental groups; however, we thought it appropriate here to develop systematically an analysis of singularities in an arbitrary two dimensional regular local domain, and also to include some related considerations for local domains; this accounts for the length of the paper. Presently we shall describe the contents of the various sections.

*Section 1.* Notations and conventions are fixed and some preliminary remarks are made.

*Section 2.* Some auxiliary lemmas, mainly on local rings, are proved. This section need not be read in the beginning; and the reader may look up the relevant parts of it when referred to in the following sections.

*Section 3.* Here is introduced the notion of an  $m$ -th quadratic transform (for any non-negative integer  $m$ ) of a regular local domain  $R$  of dimension greater than one, a quadratic transform of  $R$  is then an  $m$ -th quadratic transform of  $R$  for some  $m$ . It is proved that there is a natural one to one correspondence between the set of all quadratic transforms of  $R$  and the set of all quadratic transforms of the completion  $\bar{R}$  or  $R$ ; in this correspondence an  $m$ -th quadratic transform of  $R$  corresponds to an  $m$ -th quadratic transform of  $\bar{R}$  (Proposition 1).

*Section 4.* Let  $R$  be a two dimensional regular local domain, let  $M$  be the maximal ideal in  $R$ , let  $A$  and  $B$  be non-zero principal ideals in  $R$ , and let  $S$  be a two dimensional regular local domain having the same quotient field as  $R$  and having center  $M$  in  $R$ . The  $R$ -leading degree of  $A$  is denoted by  $\lambda_R(A)$  and is called the multiplicity of  $A$  at  $R$ . The notion of the  $S^R$  transform of  $A$  is introduced, it is denoted by  $S^R[A]$  and its  $S$ -leading degree is denoted by  $\mu_{S,R}(A)$  (Definition 3). There is a natural one to one correspondence between the immediate (i.e., first) quadratic transforms of  $R$  and irreducible forms in two variables over  $R/M$  (Lemma 13). The notion of a valuation branch of  $A$  at  $R$  is introduced, the set of all valuation branches of  $A$  at  $R$  is a finite set and it is denoted by  $\Theta(A, R)$  (Definition 4), there is a one to one natural correspondence between the valuation branches of  $A$  at  $R$  and the analytic branches of  $A$  at  $R$  (Lemma 14). If  $A$  and  $B$  are co-prime, then so are  $S^R[A]$  and  $S^R[B]$ , if  $A$  is primary then so is  $S^R[A]$ , the operation of taking transforms is transitive in  $S$  and is multiplicative in the primary factors of  $A$ , if  $\lambda_R(A) \leq 1$  then  $\mu_{S,R}(A) \leq 1$ ,<sup>2</sup> and finally  $\Theta(A, R) \supset \Theta(S^R[A], S)$  (Lemma 15). For each  $m$ , there are at most a finite number (and at least one if  $A \neq S$ ) of  $m$ -th quadratic transforms  $R_m$  of  $R$  through which  $A$  passes (i.e.,  $R_m^R[A] \neq R_m$ ) (Lemma 16). The concept of a quadratic transform of an algebraic surface is recalled (Definition 5). If  $R$  is complete and  $z$  is a non-zero irreducible non-unit in  $R$  then the reduced  $R$ -leading form of  $z$  cannot have coprime factors in  $(R/M)[X, Y]$  (Lemma 17), this is false for dimension  $> 2$  (Remark 1).

*Section 5.* Let  $R$  be a regular local domain and let  $A$  be a principal ideal in  $R$ . The notion of  $A$  to have a normal crossing at  $R$ , and the notion of  $A$  to have a strong normal crossing at  $R$  are introduced (Definition 6). Let  $S$  be a quadratic transform of  $R$ . If  $A$  has a normal crossing (respectively: strong normal crossing) at  $R$  then  $S^R[A]$  has a normal crossing

<sup>2</sup> More generally, it can be shown that always  $\mu_{S,R}(A) \leq \lambda_R(A)$ .

(respectively: a strong normal crossing) at  $S$  (Lemma 18). If  $R$  is the quotient ring of a point on an algebraic surface, if  $A$  has a normal crossing at  $R$ , and if  $S \neq R$ , then  $S^R[A]$  has a strong normal crossing at  $S$  (Lemma 19); this is false for dimension  $> 2$  (Remark 2). If  $R$  is two dimensional and is either algebraic or absolute then the singularities of  $A$  can be resolved by applying quadratic transformations to  $R$  (Proposition 2).

*Section 6.* Let  $A$  and  $B$  be non-zero principal ideals in a two dimensional regular local domain  $R$ . The strength of singularity of  $A$  on  $B$  at  $R$  is introduced and is denoted by  $\nu(A, B; R)$  (Definitions 7, 8); and via it, the notion of the strength of singularities of a curve  $W$  on another curve  $W^*$  on a non-singular algebraic surface  $V$  is introduced by taking the sum of the strengths of singularities at all the points of  $V$ , it is denoted by  $\nu(W, W^*; V)$  (Definition 9).  $\nu(A, B; R)$  is an analytic invariant and hence can be defined by using completions only (Proposition 3);  $\nu(A, B; R)$  is finite if  $A$  is a product of distinct prime ideals and  $R$  is algebraic or absolute (Proposition 4); consequently  $\nu(W, W^*; V)$  is finite (Definition 9). If  $W$  is an irreducible component of  $W^*$ , if  $\dim |W| > 1 + \nu(W, W^*; V)$ , then there exists a quadratic transform  $(V^*, f)$  of  $V$  such that  $f^{-1}(W^*)$  has only strong normal crossings on  $f^{-1}[W]$  and  $\dim |f^{-1}[W]| > 1$  (Proposition 5); this is proved by applying successive immediate quadratic transforms to  $V$  and estimating at each stage the decrease in  $\dim |W|$ .

*Section 7.* The strength of a singularity is computed for the following cases: contact of two simple branches (of arbitrary order), ordinary point (several simple branches with distinct tangents), cusps (of arbitrary order), composite cusps with distinct tangents, etc. All these notions are developed and the corresponding computations are made in the set up of an arbitrary two dimensional regular local domain. The results of this section are used only in Section 10; hence the reading of this section may be postponed until then.

*Section 8.* For dimension two, a direct proof of the abelian character of local galois groups over a branch curve having a normal crossing is given (Proposition 11 $\beta$ ) without using "Purity of branch locus"; and "Purity" is derived from it as a corollary. Most of the techniques used in the proof carry over for arbitrary two dimensional regular local domains and hence the proof can probably be generalized to this general case (Remark 4). Above a strong normal crossing of the apparent branch locus, there can be no local splitting (Proposition 12 $\beta$ ); this is in general false for normal crossings which

are not strong normal crossings (Remark 5); this necessitates the replacement of "normal crossing" by "strong normal crossing" in some of the results of Part I. This, together with other minor corrections to Part I, is given in Remark 6.

*Section 9.* Using Proposition 5 of Section 6, and then applying the technique of the proof of Proposition 6 of Section 11 of Part I, the following refinement of the quoted result of Part I is obtained: If  $W$  is a curve on a non-singular algebraic surface and  $W_1$  is an irreducible component of  $W$  such that  $\dim |W_1| > 1 + \nu(W_1, W; V)$ , if  $V^*$  is a tamely ramified covering of  $V$  and  $\phi$  is the rational map of  $V^*$  onto  $V$ , and if the branch locus of  $V^*$  over  $V$  is contained in  $W$ , then  $\phi^{-1}(W_1)$  is irreducible (Proposition 14). In view of this refinement of Proposition 6 of Part I, the main results of this section now follows by essentially carrying over the proofs of the corresponding results of Sections 11 and 12 of Part I. For dimension two, the main results of Part I are now subsumed under the results of this section.

*Section 10.* For dimension two, the results of Section 9 now give the corresponding refinements of the results of Sections 13, 14, 15 of Part I (including the Theorems of Zariski and Picard) and the latter are now subsumed under the results of this section (Propositions 17, 18, 19 and Theorems 3, 4, 5). Several explicit corollaries of the results of Sections 9 and 10 can be obtained using the computations of Sections 7, some examples of this are given (Examples 1, 2, 3).

**1. Conventions and notations.** Part I of this series with the subtitle "Branch loci with normal crossings; Applications: Theorems of Zariski and Picard" (which appeared in the *American Journal of Mathematics*, vol. 81 (1959), pp. 46-94.) will be referred to as "Part I." Besides the conventions and notations introduced in Part I we shall use the following additional ones.

In a ring  $A$ , the product over an empty set of elements will be one, thus for  $a_1, \dots, a_m$  in  $A$ :  $a_1 a_2 \cdots a_m = 1$  if  $m = 0$ ; the sum over an empty set of elements will be zero, thus for  $a_1, \dots, a_m$  in  $A$ :  $a_1 + a_2 + \cdots + a_m = 0$  if  $m = 0$ ; the product or intersection over an empty set of ideals in  $A$  will be  $A$  itself, thus for ideals  $B_1, \dots, B_m$ :  $B_1 B_2 \cdots B_m = B_1 \cap B_2 \cap \cdots \cap B_m = A$  if  $m = 0$ . For an ideal  $B$  in a ring  $A$ , by  $\text{Rad}_A B$  we shall denote the set of all elements  $a$  in  $A$  such that  $a^n$  is in  $B$  for some positive integer  $n$ ; the subscript  $A$  may be dropped when it is clear from the context.

For a valuation  $v$  of a field  $K$  we shall denote by  $R_v$  and  $M_v$ , the valuation ring of  $v$  and the maximal ideal in the valuation ring of  $v$  respectively. Let

$A$  be a ring and  $P$  an ideal in  $A$ ; if for a local ring  $(R, M)$  we have  $R \supset A$  and  $M \cap A = P$  then we shall say that  $R$  has center  $P$  in  $A$ ; if for a valuation  $v$  we have  $R_v \supset A$  and  $M_v \cap A = P$  then we shall say that  $v$  has center  $P$  in  $A$ . If  $(R, M)$  is a local domain and  $v$  is a valuation having center  $M$  in  $R$ , then the transcendence degree of  $R_v/M_v$  over  $R/M$  will be called the  $R$ -dimension of  $v$ . If  $A$  is a ring and  $P$  is a finitely generated ideal in  $A$ , and  $v$  is a valuation with  $R_v \supset R$ , then by  $v(P)$  we shall denote the minimum of  $v(a)$  for  $a$  in  $P$ ; note that if  $Q$  is a set of generators of  $P$  then  $v(P)$  is the minimum of  $v(a)$  for  $a$  in  $Q$ .

A local ring  $(R, M)$  is said to be regular, if and only if,  $R$  is noetherian, and  $M$  has a basis of  $n$  elements, where  $n$  is the dimension of  $R$ . Note that a regular local domain  $R$  of dimension two is a unique factorization domain, and hence in  $R$  every pure one dimensional ideal is principal, products and intersections of pure one dimensional ideals are the same things, etc.

Now let  $(R, M)$  be a regular local domain. For a non-zero ideal  $A$  in  $R$  there exists a unique integer such that  $A \subset M^n$  and  $A \not\subset M^{n+1}$ , we shall call  $n$  the  $R$ -leading degree of  $A$  or the multiplicity of  $A$  at  $R$  and we shall denote it by  $\lambda_R(A)$ ; similarly, for a non zero element  $a$  of  $R$  the unique integer  $n$ , for which  $a \in M^n$  and  $a \notin M^{n+1}$ , will be called the  $R$ -leading degree of  $a$  or the multiplicity of  $a$  at  $R$  and will be denoted by  $\lambda_R(a)$ ; note that  $\lambda_R(a) = \lambda_R(aR)$ . Let  $Q$  be a representative set of  $R/M$  in  $R$  (i. e., a subset of  $R$  which is mapped one to one onto  $R/M$  under the canonical homomorphism of  $R$  onto  $R/M$ ), and let  $x = (x_1, \dots, x_n)$  be a minimal basis of  $M$ . Then, for  $0 \neq a \in R$  there exists a unique form  $f(X_1, \dots, X_n)$  of degree  $\lambda_R(a)$  in the indeterminates  $X_1, \dots, X_n$  with coefficients in  $Q$  such that  $a - f(x_1, \dots, x_n)$  is in  $M^{\lambda_R(a)+1}$  and we shall call  $f(X_1, \dots, X_n)$  the  $R$ -leading form of  $a$  with respect to  $(Q, x)$  and we shall denote it by  $\Delta_{R, Q, x}(a)$ ; furthermore, the form obtained from  $f(X_1, \dots, X_n)$  by reducing its coefficients modulo  $M$  will be called the reduced  $R$ -leading form of  $a$  with respect to  $x$  and will be denoted by  $\bar{\Delta}_{R, x}(a)$ ; for  $a = 0$  we set  $\Delta_{R, Q, x}(a) = \bar{\Delta}_{R, x}(a) = 0$ . Again, one or more subscripts of  $\lambda, \Delta, \bar{\Delta}$  may be dropped when they are clear from the context. Note that for  $a_1, \dots, a_m$  in  $R$  we have  $\bar{\Delta}(a_1 \cdots a_m) = \bar{\Delta}(a_1) \cdots \bar{\Delta}(a_m)$ . Also note that, if  $y = (y_1, \dots, y_n)$  is any other minimal basis of  $M$  and if  $f(X_1, \dots, X_n) = \bar{\Delta}_x(a)$  and  $g(X_1, \dots, X_n) = \bar{\Delta}_y(a)$ , then  $g$  is obtained from  $f$  by applying a non-singular homogeneous linear transformation to  $X_1, \dots, X_n$  with coefficients in  $R/M$ ; and hence if  $f = f_1^{u_1} \cdots f_s^{u_s}$  and  $g = g_1^{v_1} \cdots g_t^{v_t}$  are factorizations of  $f$  and  $g$  where  $f_1, \dots, f_s$  are pair-wise co-prime irreducible forms and  $g_1, \dots, g_t$  are pair-wise co-prime irreducible forms and  $u_i, v_i > 0$ , then

$s = t$  and after a suitable relabelling  $u_i = v_i$ , and  $f_i$  and  $g_i$  are non-zero constant multiples of each other for  $i = 1, \dots, s$ .

Now let  $x_2, \dots, x_n$  be elements of  $R$ , such that there exists  $x_1$  in  $R$  such that  $x_1, \dots, x_n$  is a basis of  $M$ ; if for the reduced  $R$ -leading form  $f(X_1, \dots, X_n)$  of  $a$  with respect to  $x_1, \dots, x_n$  we have  $f(1, 0, 0, \dots, 0) \neq 0$ , then we shall say that "the line  $x_2 = \dots = x_n = 0$  is non-tangential to  $a$ "; note that this does not depend on  $x_1$ ; also note that if  $a_1, \dots, a_t$  are elements of  $R$ , then the line  $x_2 = \dots = x_n = 0$  is non-tangential to  $a_1 \dots a_n$ , if and only if, it is non-tangential to each  $a_i$ .

Next let  $V$  be an  $n$ -dimensional irreducible projective algebraic variety over an algebraically closed ground field  $k$ . If  $W$  is an irreducible subvariety of  $V$ , then  $P \in W$  if and only if  $Q(W, V) \supset Q(P, V)$ , and in that case we set  $M(P, W, V) = M(W, V) \cap Q(P, V)$ ; if  $P \notin W$ , then we set  $M(P, W, V) = Q(P, V)$ . If  $W$  is any subvariety of  $V$  with irreducible components  $W_1, \dots, W_t$ , then we shall call  $M(P, W_1, V) \cap \dots \cap M(P, W_t, V)$  the ideal of  $W$  at  $P$  on  $V$  and we shall denote it by  $M(P, W, V)$ ; note that  $P \in W$  if and only if  $M(P, W, V) \subset M(P, V)$ ; an ideal  $A$  in  $Q(P, V)$  will be called a defining ideal of  $W$  at  $P$  on  $V$  if  $A = A_1 \cap \dots \cap A_t$  where  $A_i$  is an ideal in  $Q(P, V)$  which is primary for  $M(P, W_i, V)$ ; note that then  $A_1, \dots, A_t$  are uniquely determined by  $A$ ; now assume that  $P$  is a simple point of  $V$  and that  $W_1, \dots, W_t$  are all of co-dimension 1, then: (1) a defining ideal of  $W$  at  $P$  on  $V$  is exactly a principal ideal  $A$  in  $Q(P, V)$  such that  $\text{Rad } A = M(P, W, V)$ ; (2) a given ideal  $A$  in  $Q(P, V)$  is a defining ideal of a pure one co-dimensional subvariety of  $V$  if and only if  $A$  is a non-zero principal ideal; and (3) a given ideal  $A$  in  $Q(P, V)$  is the ideal of a pure one co-dimensional subvariety of  $V$  if and only if  $A$  is a non-zero principal ideal which is not contained in the square of any non-unit principal ideal.

If  $V^*$  and  $V$  are irreducible normal projective algebraic varieties over an algebraically closed ground field  $k$  and  $f$  is a bi-rational map of  $V^*$  onto  $V$  which is regular on  $V^*$ , and if  $W$  is a subvariety of  $V$ , then by  $f^{-1}[W]$  we shall denote the transform of  $W$  under  $f^{-1}$  (for definition see [Zariski 15]), and by  $f^{-1}(W)$  we shall denote as usual the set theoretic inverse image of the set of points of  $W$ . It can immediately be verified that if  $W$  is irreducible and of co-dimension one so that  $Q(W, V)$  is the valuation ring of a real discrete valuation  $v$  of  $k(V)/k$ , then  $f^{-1}[W]$  is also irreducible of co-dimension one and it is the center of  $v$  on  $V^*$ .

## 2. Auxiliary lemmas.

LEMMA 1. Let  $(R, M)$  be a regular local domain of dimension  $n > 1$  with quotient field  $K$ . Let  $x_1, \dots, x_n$  be a minimal basis of  $M$ . Let  $y_1 = x_1$ ,  $y_i = x_i/x_1$  for  $i = 2, \dots, n$  and let  $S = R[y_2, \dots, y_n]$ . Then we have  $(y_1 S) \cap R = M$  so that  $R/M$  can be canonically identified with a subfield of  $S/(y_1 S)$  and then the residues  $\bar{y}_2, \dots, \bar{y}_n$  respectively of  $y_2, \dots, y_n$  modulo  $y_1 S$  are algebraically independent over  $R/M$  and hence  $S/(y_1 S)$  can be identified with the polynomial ring  $(R/M)[\bar{y}_2, \dots, \bar{y}_n]$  in  $n-1$  variables. Then the canonical homomorphism of  $S$  onto  $S/(y_1 S)$  sets up a one to one correspondence between the maximal ideals in  $S$  containing  $y_1$  and the maximal ideals in the polynomial ring  $S/(y_1 S)$ , and hence if  $P$  is a maximal ideal in  $S$  containing  $y_1$  and  $R_1 = S_P \subset K$  and  $M_1 = PR_1$  then  $(R_1, M_1)$  is a regular local domain of dimension  $n$  having center  $M$  in  $R$ ,  $y_1$  is part of a minimal basis of  $M_1$ , and  $R_1/M_1$  is a finite algebraic extension of  $R/M$ .

*Proof.* Follows from [Abhyankar 5, Lemma 3.19 of Section 14].

LEMMA 2. Let  $(R, M)$  and  $(\bar{R}, \bar{M})$  be two regular local domains of the same dimension  $n$  with quotient fields  $K$  and  $\bar{K}$  respectively. Assume that  $K$  is a subfield of  $\bar{K}$ ,  $\bar{R}$  has center  $M$  in  $R$  and  $M\bar{R} = \bar{M}$ . Then  $R/M$  can be canonically identified with a subfield of  $\bar{R}/\bar{M}$ ; assume that under this identification  $\bar{R}/\bar{M} = R/M$ , i. e., given  $x$  in  $\bar{R}$  there exists  $y$  in  $R$  such that  $x - y$  is in  $\bar{M}$ . Then given  $x$  in  $R$  and a positive integer  $q$ , there exists  $y$  in  $R$  such that  $x - y$  is in  $\bar{M}^q$ . Also we have that  $K \cap \bar{R} = R$ ,  $K \cap \bar{M} = M$ ,  $M^q R = \bar{M}^q$  for any positive integer  $q$ , and if  $A$  is any ideal in  $R$  then  $(A\bar{R}) \cap R = A$ , and  $A$  is principal if and only if  $A\bar{R}$  is principal, in particular  $\bar{M}^q \cap R = M^q$  for any positive integer  $q$ . Consequently, any completion of  $\bar{R}$  is also a completion of  $R$ .

*Proof.* From  $M\bar{R} = \bar{M}$  we at once have  $M^q \bar{R} = \bar{M}^q$  for all  $q > 0$ . By induction on  $q$  we shall show that given  $x$  in  $\bar{R}$  there exists  $y$  in  $R$  such that  $x - y$  is in  $\bar{M}^q$ ; for  $q = 1$  this is given, so assume that  $q > 1$  and that this is true for  $q - 1$ . Then given  $x$  in  $\bar{R}$  there exists  $z$  in  $R$  such that  $x - z$  is in  $\bar{M}^{q-1}$ ; since  $\bar{M}^{q-1} = M^{q-1} \bar{R}$ ,  $x - z = \sum a_i b_i$  with  $a_i$  in  $M^{q-1}$  and  $b_i$  in  $\bar{R}$ , the assumption for  $q = 1$  tells us that there exists  $c_i$  in  $R$  and  $d_i$  in  $\bar{M}$  such that  $b_i = c_i + d_i$ , let  $y = z + \sum a_i c_i$ , then  $x - y = \sum a_i d_i$  which is in  $\bar{M}^q$ .

Now let  $x_1, \dots, x_n$  be a minimal basis of  $M$ . Since  $M\bar{R} = \bar{M}$  and  $\dim \bar{R} = n$ ;  $x_1, \dots, x_n$  is also a minimal basis of  $\bar{M}$ . For any non-negative integer  $q$ ,  $y \in \bar{M}^q$  and  $y \notin \bar{M}^{q+1}$  imply that  $y = f(x_1, \dots, x_n)$  where  $f(X_1, \dots, X_n)$  is a form of degree  $q$  with coefficients in  $R$  but not all in  $M$ ; suppose if possible

that  $y \in \bar{M}^{q+1}$ , then  $y = g(x_1, \dots, x_n)$  where  $g(X_1, \dots, X_n)$  is a form of degree  $q+1$  with coefficients in  $\bar{R}$ , so that  $y = h(x_1, \dots, x_n)$  where  $h(X_1, \dots, X_n)$  is a form of degree  $q$  with coefficients in  $\bar{M}$ ; let  $t(X_1, \dots, X_n) = f(X_1, \dots, X_n) - h(X_1, \dots, X_n)$ ; since  $f$  has coefficients in  $R$  but not all in  $M$  and since  $\bar{M} \cap R = M$ , these coefficients are in  $\bar{R}$  but not all in  $\bar{M}$ ; since the coefficients of  $h$  are all in  $\bar{M}$  we conclude that  $t(X_1, \dots, X_n)$  is a form of degree  $q$  with coefficients in  $\bar{R}$  but not all in  $\bar{M}$  and hence  $t(x_1, \dots, x_n) \notin \bar{M}^{q+1}$  so that  $t(x_1, \dots, x_n) \neq 0$  which is a contradiction since  $t(x_1, \dots, x_n) = y - y = 0$ ; consequently,  $y \in M^q$  and  $y \notin \bar{M}^{q+1}$  imply that  $y \notin \bar{M}^{q+1}$ . Therefore, for any positive integer  $q$ ,  $y \in R$  and  $y \notin M^q$  imply that  $y \notin \bar{M}^q$ ; hence  $y \in R$  and  $y \in \bar{M}^q$  imply that  $y \in M^q$ . Therefore  $R \cap \bar{M}^q \subset M^q$ , and hence  $R \cap \bar{M}^q = M^q$ .

Next, let  $A$  be any ideal in  $R$ . Let  $y \in (A\bar{R}) \cap R$  and let  $q$  be any positive integer; then  $y \in A\bar{R}$  implies that  $y = \sum a_i z_i$  with  $a_i$  in  $A$  and  $z_i$  in  $\bar{R}$  so that  $z_i = u_i + t_i$  with  $u_i$  in  $R$  and  $t_i$  in  $\bar{M}^q$ , hence  $y = \sum a_i u_i + \sum a_i t_i$  with  $\sum a_i u_i$  in  $A$  and  $\sum a_i t_i$  in  $\bar{M}^q$ ; since  $\sum a_i t_i = y - \sum a_i u_i$  with  $y$  in  $R$  and  $\sum a_i u_i$  in  $R$  we have  $\sum a_i t_i \in R \cap \bar{M}^q = M^q$  so that  $y = \sum a_i u_i + \sum a_i t_i \in A + M^q$ .

Thus  $(A\bar{R}) \cap R \subset \bigcap_{q=1}^{\infty} (A + M^q) = A$ , and hence  $(A\bar{R}) \cap R = A$ . Now it is obvious that if  $A$  is principal then so is  $A\bar{R}$ , so assume conversely that  $A\bar{R}$  is principal and let  $\bar{y}$  be a generator of  $A\bar{R}$ . Then it is obvious that if  $\bar{y} = 0$  then  $A = 0$ , so assume that  $\bar{y} \neq 0$  and let  $q$  be the unique integer for which  $\bar{y} \in \bar{M}^q$  and  $\bar{y} \notin \bar{M}^{q+1}$  so that  $A\bar{R} \subset \bar{M}^q$  and  $A\bar{R} \not\subset \bar{M}^{q+1}$ . Since  $\bar{M}^{q+1}\bar{R} = \bar{M}^{q+1}$ , there exists  $y$  in  $A$  such that  $y \notin \bar{M}^{q+1}$ . Since  $\bar{M}^{q+1} \cap R = \bar{M}^{q+1}$ ,  $y \notin \bar{M}^{q+1}$ . Now  $y = \bar{y}z$  with  $z$  in  $\bar{R}$ . Since  $\bar{y} \in \bar{M}^q$  and  $y \notin \bar{M}^{q+1}$ ,  $z$  must be a unit in  $\bar{R}$  and hence  $A\bar{R} = y\bar{R}$  so that  $A = (A\bar{R}) \cap R = ((y\bar{R})\bar{R}) \cap R = yR$ .

Finally, take  $y$  in  $K \cap \bar{R}$ , then  $y = z/t$  with  $z \in R$  and  $0 \neq t \in R$ , hence  $z = ty \in (t\bar{R}) \cap R = tR$ , and hence  $y = z/t \in R$ . Therefore  $K \cap \bar{R} \subset R$ , and hence  $K \cap \bar{R} = R$ , and  $K \cap \bar{M} = (K \cap \bar{M}) \cap \bar{R} = (K \cap \bar{R}) \cap \bar{M} = R \cap \bar{M} = M$ .

**LEMMA 3.** (*A form Weierstrass Preparation Theorem*) Let  $R$  be the formal power series ring  $k[[x_1, \dots, x_n]]$  in  $x_1, \dots, x_n$  ( $n > 1$ ) with coefficients in a field  $k$ , and let  $M$  be the maximal ideal in  $R$ . Let  $L$  be "the line  $x_2 = \dots = x_n = 0$ ." Let  $S = k[[x_2, \dots, x_n]]$ , and let  $N$  be the maximal ideal in  $S$ . If  $y$  is a nonzero element in  $R$  to which  $L$  is non-tangential, then there exists a unique unit  $y^*$  in  $R$  such that  $y = y'y^*$ , where  $y' = x_1^b + y_1x_1^{b-1} + \dots + y_b$ ,  $y_i \in S$ ,  $b = \lambda_R(y)$ ; furthermore  $y_1, \dots, y_b$  are necessarily in  $N$ . If  $z$  is a nonzero element in  $S[x_1]$ ,  $t$  is a non-unit factor of  $z$  in  $R$  such that  $L$  is non-tangential to  $t$ ,  $t^*$  is the unique unit in  $R$  such that  $t = t't^*$ , where

$l' = x_1^d + t_1 x_1^{d-1} + \cdots + t_d$ ,  $t_i \in S$ ,  $d = \lambda_R(t)$ , then  $L$  is non-tangential to  $z$  and  $l'$  divides  $z$  in  $S[x_1]$ . Finally, if  $e$  is a non-zero element of  $R$ , and  $u_1, \cdots, u_n$  is a basis of  $M$ , and either  $k$  is infinite or  $\lambda_R(e) \leq 2 = n$ , then there exist elements  $a_{ij}$  in  $k$  such that letting  $v_i = \sum a_{ij} u_j$ , we have that  $v_1, \cdots, v_n$  is a basis of  $M$  and the line  $v_2 = \cdots = v_n = 0$  is non-tangential to  $e$ .

*Proof.* Follows from well known considerations; see for instance Expose X in H. Cartan's Seminar of 1951-1952. The final statement follows from the facts that the projective plane over any field has at least three rational points, and that in a projective space over an infinite field there exist rational points lying outside a given hyper-surface.

**LEMMA 4.** Let  $(R, M)$  be a one dimensional noetherian local domain with quotient field  $K$ . Then the integral closure  $S$  of  $R$  in  $K$  is a principal ideal domain with only a finite number of prime ideals  $P_1, \cdots, P_n$  ( $n > 0$ ), so that  $S$  is the intersection of  $R_{v_1}, \cdots, R_{v_n}$  where  $v_i$  is the real discrete valuation of  $K$  with  $R_{v_i} = S_{P_i}$ , each  $v_i$  has center  $M$  in  $R$  and  $v_1, \cdots, v_n$  are the only valuations of  $K$  whose valuation rings contain  $R$ . Furthermore, if  $R$  is complete then  $n = 1$ .<sup>a</sup>

*Proof.* Everything except the last statement follows from [Krull 10, Section 39]. Now assume that  $R$  is complete. Let  $a$  be any element of  $P_1$ , and let  $f(X) = X^m + f_1 X^{m-1} + \cdots + f_m$  be the monic polynomial of least degree with coefficients  $f_i$  in  $R$  such that  $f(a) = 0$ . Then  $a \in P_1$  and  $f_i \in R \subset S$  for  $i = 1, \cdots, m$  imply that  $f_m \in P_1 \cap R = P$ . If some  $f_i$  were not in  $M$ , then in view of the fact that  $S$  has no non-zero zero-divisors, by Hensel's Lemma applied to the monic polynomial  $f(X)$  over the complete local domain  $R$  we would get a monic polynomial  $g(X)$  in  $R[X]$  of degree smaller than  $m$  for which  $g(a) = 0$ , whence  $f_i \in M$  for all  $i$ . For any  $j$ ,  $P_j \supset P$  and hence  $f_i \in P_j$  for all  $i$ ; consequently  $a^m \in P_j$ , and hence  $a \in P_j$ . Therefore  $P_1 \subset P_j$  for all  $j$ , and hence  $n = 1$ .

**LEMMA 5.** Let  $(R, M)$  be  $n$ -dimensional regular local domain such that  $R$  contains a coefficient field  $k$ , and let  $s$  be a non negative integer. Then  $R/M^s$  can, in a natural way, be considered to be a vector space over  $k$ , and then its  $k$ -dimension is  $n(s) = \binom{n+s-1}{n}$ , i.e., (i) there exist elements  $y_1, \cdots, y_{n(s)}$  in  $R$  such that, if  $c_1, \cdots, c_{n(s)}$  are elements in  $k$  with

<sup>a</sup> The proof given below yields that, more generally, without the assumption of  $R$  being one dimensional, if  $R$  is complete then  $S$  has a unique maximal ideal, i.e.,  $S$  is a local ring.

$c_1 y_1 + \cdots + c_{n(s)} y_{n(s)} \in M^s$  then  $c_1 = \cdots = c_{n(s)} = 0$ ; and (ii) if  $z_1, \cdots, z_t$  are elements in  $R$  with  $t > n(s)$ , then there exist  $c_1, \cdots, c_t$  in  $k$ , which are not all zero, such that  $c_1 z_1 + \cdots + c_t z_t$  is in  $M^s$ .

*Proof.* Since  $k \subset R$ ,  $R$  as well as  $M^s$  and hence  $R/M^s$  can be considered as vector spaces over  $k$ . Fix a minimal basis  $x_1, \cdots, x_n$  of  $M$  and let  $y_1, \cdots, y_q$  be all distinct monomials in  $x_1, \cdots, x_n$  of degree  $< s$ . Then  $q = n(s)$  and by well known properties of regular local domains it follows that  $y_1, \cdots, y_q$  form a  $k$ -basis of  $R$  modulo  $M^s$ .

LEMMA 6. Let  $V$  and  $V^*$  be normal irreducible projective algebraic varieties over an algebraically closed ground field  $k$ , let  $\phi$  be a regular birational map of  $V^*$  onto  $V$ , and let  $K'$  be a finite separable extension of  $k(V) = k(V^*)$ . If  $K'/V$  is tamely ramified then so is  $K'/V^*$ .

*Proof.* Follows from Lemmas 6 and 7 of Part I and [Abhyankar 5, Proposition 1.50 of Section 7].

LEMMA 7. Let  $R$  be a subdomain of a field  $K$ , let  $x_1, \cdots, x_n$  be elements in  $R$  with  $x_1 \neq 0$ , let  $M$  be the ideal in  $R$  generated by  $x_1, \cdots, x_n$ ; let  $S = R[x_2/x_1, \cdots, x_n/x_1]$  and let  $a$  be any element of  $K$ . Then  $a \in S$  if and only if there exists  $q \geq 0$  such that  $x_1^q a \in M^q$ , and  $a \in x_1 S$  if and only if there exists  $q \geq 0$  such that  $x_1^q a \in M^{q+1}$ .

*Proof.* The statements being obvious for  $n=1$  we shall take  $n > 1$ . First assume that  $a \in S$ . Then  $a = f(x_2/x_1, \cdots, x_n/x_1)$  with  $f(Y_2, \cdots, Y_n) \in R[Y_2, \cdots, Y_n]$  so that  $f(x_2/x_1, \cdots, x_n/x_1) = F(x_1, \cdots, x_n)/x_1^q$ , where  $F(X_1, \cdots, X_n)$  is a form of some degree  $q \geq 0$  with coefficients in  $R$ , and we have  $x_1^q a = F(x_1, \cdots, x_n) \in M^q$ . Conversely, assume that  $x_1^q a \in M^q$  for some  $q \geq 0$ . Then  $x_1^q a = F(x_1, \cdots, x_n)$  where  $F(X_1, \cdots, X_n)$  is a form of degree  $q$  with coefficients in  $R$  and hence  $a = F(x_1, \cdots, x_n)/x_1^q = F(1, x_2/x_1, \cdots, x_n/x_1) \in S$ . Now assume that  $a \in x_1 S$ . Then  $a = x_1 b$  with  $b \in S$  so that  $x_1^q b \in M^q$  for some  $q \geq 0$ , and hence  $x_1^q a = x_1(x_1^q b) \in M^{q+1}$ . Conversely assume that  $x_1^q a \in M^{q+1}$  for some  $q \geq 0$ . Then  $x_1^q a = F(x_1, \cdots, x_n)$  where  $F(X_1, \cdots, X_n)$  is a form of degree  $q+1$  with coefficients in  $R$  and hence  $a/x_1 = F(x_1, \cdots, x_n)/x_1^{q+1} = F(1, x_2/x_1, \cdots, x_n/x_1) \in S$  so that  $a \in x_1 S$ .

LEMMA 8. Let  $G$  and  $H$  be finite cyclic groups and let  $\alpha: G \rightarrow H$  be an epimorphism. Let  $h$  be a generator of  $H$ . Then there exists a generator  $g$  of  $G$  for which  $\alpha(g) = h$ .

*Proof.* Let  $m$  be the order of  $G$  and let  $n$  be the order of  $H$ . Then  $n$

divides  $m$ . Let  $r$  be a generator of  $G$ . Then for some integer  $u$ ,  $\alpha(r^u) = h$ . Let  $p_1, \dots, p_a, q_1, \dots, q_b$  be the distinct prime divisors of  $m$  so labelled that  $p_1, \dots, p_a$  do not divide  $u$ , and  $q_1, \dots, q_b$  do divide  $u$ . Let  $v = np_1 \dots p_a$ , let  $w = u + v$ , and let  $g = r^w$ . Since  $r$  is a generator of  $G$ ,  $\alpha(r)$  is a generator of  $H$ . Since  $h = \alpha(r^u) = (\alpha(r))^u$  is also a generator of  $H$  and since  $H$  is of order  $n$ , it follows that  $u$  and  $n$  are co-prime and hence  $q_1, \dots, q_b$  do not divide  $n$ , which implies that they do not divide  $v$ , and hence they do not divide  $w$ . Also, any  $p_i$  does not divide  $u$  but does divide  $v$ , and hence does not divide  $w$ . Therefore, none of the prime factors  $p_1, \dots, p_a, q_1, \dots, q_b$  of  $m$  divide  $w$ , and hence  $g$  is a generator of  $G$ . Since  $H$  is of order  $n$ , and  $n$  divides  $v$ , we have  $\alpha(r)^v = 1$  and hence  $\alpha(g) = \alpha(r)^{u+v} = \alpha(r)^u = h$ .

**3. Quadratic transforms of regular local domains.** Throughout this section,  $(R, M)$  will denote a regular local domain of dimension  $n > 1$ , with quotient field  $K$ .

**DEFINITION 1.** A local domain  $(R_1, M_1)$  with quotient field  $K$  will be called an immediate or a first quadratic transform of  $R$  if there exists  $x \in M$ ,  $x \notin M^2$  such that  $R_1$  is the quotient ring in  $K$  of  $R[Mx^{-1}]$  with respect to a maximal ideal in  $R[Mx^{-1}]$  containing  $x$ , where  $R[Mx^{-1}]$  denotes the ring obtained by adjoining to  $R$  all the elements in  $K$  of the form  $yx^{-1}$  with  $y$  in  $M$ ; now  $x \in M$  and  $x \notin M^2$  implies that there exists a minimal basis  $x = x_1, x_2, \dots, x_n$  of  $M$  and then  $R[Mx^{-1}] = R[x_2/x_1, \dots, x_n/x_1]$ , and hence by Lemma 1,  $R_1$  is a regular local domain of dimension  $n$  having center  $M$  in  $R$ . By induction we define an  $m$ -th quadratic transform of  $R$  to be an immediate quadratic transform of an  $(m-1)$ -st quadratic transform of  $R$ . Also we shall call  $R$  its only 0-th quadratic transform. Finally if  $R^*$  is an  $m$ -th quadratic transform of  $R$  for some non-negative integer  $m$ , then we shall say that  $R^*$  is a quadratic transform of  $R$ .

**LEMMA 9.** Let  $(R^*, M^*)$  be an immediate quadratic transform of  $R$ , and let  $z_1, \dots, z_n$  be a minimal basis of  $M$ . Then there exists  $i$  such that  $z_i/z_i \in R^*$  for  $j = 1, \dots, n$ ; and if we choose any one such value of  $i$ , then setting  $S = R[z_1/z_i, \dots, z_n/z_i]$  we have that  $R^* \supset S$ ,  $N = S \cap M^*$  is a maximal ideal in  $S$  containing  $z_i$ ,  $R^* = S_N$ ,  $M^* = NR^*$ ,  $MR^* = z_i R^*$ , and  $z_i$  is part of a minimal basis of  $M^*$ . Also  $R^* \neq R$ .

*Proof.* By definition there exists  $x \in M$ ,  $x \notin M^2$  such that  $R^*$  is the quotient ring of  $R[Mx^{-1}]$  with respect to a maximal ideal containing  $x$ . Let  $v$  be a valuation of  $K$  having center  $M^*$  in  $R^*$ . Then  $z_i/x \in R^*$  implies that

$v(z_j/x) \geq 0$  for all  $j$ , and hence  $v(x) = v(M)$ . Since  $z_1, \dots, z_n$  is a basis of  $M$ , there exists  $i$  such that  $v(z_i) = v(M) = v(x)$ , and then  $v(z_i/x) = 0$ ; and this, in view of the fact that  $z_i/x$  is in  $R^*$ , implies that  $z_i/x$  is a unit in  $R^*$ , and hence  $x/z_i$  is in  $R^*$ , which in turn implies that  $z_j/z_i = (z_j/x)(x/z_i)$  is in  $R^*$  for all  $j$ . Now fix  $i$  such that  $z_j/z_i$  is in  $R^*$  for all  $j$ , and let  $S = R[z_1/z_i, \dots, z_n/z_i]$ . Then  $R^* \supset S$ . Let  $N = S \cap M^*$ . Then canonically  $R^*/M^* \supset S/N \supset R/M$ ; since by Lemma 1,  $R^*/M^*$  is algebraic over  $R/M$ ,  $S/N$  must be algebraic over  $R/M$ . Since  $z_i \in M^* \cap S = N$ , by Lemma 1 we conclude that  $N$  is a maximal ideal in  $S$ . Let  $R_1 = S_N$  and  $M_1 = NS$ . Then  $R^*$  has center  $M_1$  in  $R_1$ . Now  $v$  also has center  $M_1$  in  $R_1$ , and  $x/z_i$  is in  $R_1$ , and  $v(x/z_i) = 0$ ; whence  $x/z_i$  is a unit in  $R_1$ . Hence  $R[Mx^{-1}] \subset R_1$ , and

$$\begin{aligned} M^* \cap R[Mx^{-1}] &= (M_v \cap R^*) \cap R[Mx^{-1}] \\ &= M_v \cap R[Mx^{-1}] = (M_v \cap R_1) \cap R[Mx^{-1}] = M_1 \cap R[Mx^{-1}]. \end{aligned}$$

Therefore  $R_1 = R^*$ . By Lemma 1,  $S/z_i S$  is a polynomial ring, over a field, in  $n-1 > 0$  variables, and hence contains infinitely many maximal ideals; therefore  $S$  is not a local ring. Hence  $R^* \not\subset S$  and therefore  $R^* \not\subset R$ . Since  $z_j = (z_j/z_i)z_i$  and  $(z_j/z_i) \in R^*$ ,  $MR^* = (z_1, \dots, z_n)R^* = z_i R^*$ , and by Lemma 1,  $z_i$  is part of a minimal basis of  $M^*$ .

**LEMMA 10.** *Let  $(R_1, M_1)$  and  $(R', M')$  be two distinct immediate quadratic transforms of  $R$ . Then there exists a minimal basis  $x_1, \dots, x_n$  of  $M$  such that  $x_j/x_1 \in R_1 \cap R'$  for all  $j$ . Also, any given valuation of  $K$  can have center at the maximal ideal of at most one immediate quadratic transform of  $R$ .*

*Proof.* Let  $y_1, \dots, y_n$  be a minimal basis of  $M$ , let  $v$  be a valuation of  $K$  having center  $M_1$  in  $R_1$ , and let  $v'$  be a valuation of  $K$  having center  $M'$  in  $R'$ . If for some  $i$ ,  $v(y_i) = v(M)$  and  $v'(y_i) = v'(M)$ , then, relabelling the  $y_j$  so that  $y_i = y_1$  and taking  $x_j = y_j$  for all  $j$ , we are through. In the contrary case, we can relabel the  $y_j$  so that  $v(y_1) = v(M) \neq v(y_2)$ , and  $v'(y_2) = v'(M) \neq v'(y_1)$ . Let  $x_1 = y_1 + y_2$  and  $x_j = y_j$  for all  $j > 1$ . Then  $v(x_1) = v(M)$  and  $v'(x_1) = v'(M)$ , and thus the first assertion is proved. Now suppose if possible that a valuation  $w$  of  $K$  has center  $M_1$  in  $R_1$  and center  $M'$  in  $R'$ . Let  $S = [x_2/x_1, \dots, x_n/x_1]$ . Then by Lemma 9,  $S \cap M_1$  and  $S \cap M'$  are distinct maximal ideals in  $S$ . However  $S \cap M_1 = S \cap M_w = S \cap M'$ , which is a contradiction. Hence the second assertion is proved.

**LEMMA 11.** *Let  $(R^*, M^*)$  be a quadratic transform of  $R$ . Then there exists a unique integer  $m$  such that  $R^*$  is an  $m$ -th quadratic transform of  $R$ ;*

also there exists a unique sequence  $R = R_0, R_1, \dots, R_m = R^*$  such that  $R_i$  is an immediate quadratic transform of  $R_{i-1}$  for  $i = 1, \dots, m$ . Furthermore  $R_0, \dots, R_m$  are the only quadratic transforms of  $R$  which are contained in  $R^*$ . Also, if  $v$  is a valuation of  $K$  having center  $M^*$  in  $R^*$ , then  $R_0, \dots, R_m$  are the only  $q$ -th quadratic transforms of  $R$  with  $q \leq m$ , which are contained in  $R_v$  (it is obvious that in each of them  $v$  has center at the maximal ideal).

*Proof.* Follows from Lemmas 9 and 10 by induction.

**DEFINITION 2.** Let  $v$  be a valuation of  $K$  having center  $M$  in  $R$  and of  $R$ -dimension zero. Then by Lemma 11 above and [Abhyankar 4, Lemma 10] it follows that for each  $m$  there exists a unique  $m$ -th quadratic transform  $(R_m, M_m)$  of  $R$  such that  $v$  has center  $M_m$  in  $R_m$ ; and then for all  $m$ ,  $R_m$  is an immediate quadratic transform of  $R_{m-1}$ , and the  $R_m$ -dimension of  $v$  is zero. We shall say that  $R_m$  is the  $m$ -th quadratic transform of  $R$  along  $v$  and that  $R = R_0, R_1, R_2, \dots$  is the quadratic sequence of  $R$  along  $v$ .

**LEMMA 12.** Let  $(\bar{R}, \bar{M})$  be an  $n$ -dimensional regular local domain such that the quotient field  $\bar{K}$  of  $\bar{R}$  contains  $K$  as a subfield,  $\bar{R}$  has center  $\bar{M}$  in  $R$ ,  $\bar{M}\bar{R} = \bar{M}$ , and the natural monomorphism  $\alpha: R/\bar{M} \rightarrow \bar{R}/\bar{M}$  is an epimorphism. For an immediate quadratic transform  $(\bar{R}_1, \bar{M}_1)$  of  $\bar{R}$  set  $\tau\bar{R}_1 = \bar{R}_1 \cap K$  and  $\tau\bar{M}_1 = \bar{M}_1 \cap K$ ; then  $\bar{R}_1$  has center  $\tau\bar{M}_1$  in  $\tau\bar{R}_1$ ,  $(\tau\bar{M}_1)\bar{R}_1 = \bar{M}_1$ , and the natural monomorphism  $\tau\bar{R}_1/\tau\bar{M}_1 \rightarrow \bar{R}_1/\bar{M}_1$  is an epimorphism. Furthermore,  $\tau$  maps the set of all immediate quadratic transforms of  $\bar{R}$  in a one to one manner onto the set of all immediate quadratic transforms of  $R$ .

*Proof.* Let  $x_1, \dots, x_n$  be a minimal basis of  $M$ , then it is also a minimal basis of  $\bar{M}$ . Let

$$S = R[x_2/x_1, \dots, x_n/x_1] \text{ and } \bar{S} = \bar{R}[x_2/x_1, \dots, x_n/x_1].$$

First observe that  $(x_1\bar{S}) \cap S = x_1S$ ; it being obvious that  $(x_1\bar{S}) \cap S \supset x_1S$  it is enough to show that  $(x_1\bar{S}) \cap S \subset x_1S$ , so let  $y \in (x_1\bar{S}) \cap S$ ; then by Lemma 7 there exist non-negative integers  $q$  and  $t$  such that  $x_1^q y \in \bar{M}^{q+1}$  and  $x_1^t y \in M^t$ , now  $x_1^t y \in M^t$  implies that  $x_1^t y \in R$  and hence  $x_1^{t+q} y \in R$ , also  $x_1^q y \in \bar{M}^{q+1}$  and  $x_1^t \in \bar{M}^t$  imply that  $x_1^{q+t} y \in \bar{M}^{q+t+1}$  and hence by Lemma 2,

$$x_1^{q+t} y \in \bar{M}^{q+t+1} \cap R = M^{q+t+1}$$

and therefore by Lemma 7,  $y \in x_1S$ . Therefore there is a natural monomorphism  $\beta: S/x_1S \rightarrow \bar{S}/x_1\bar{S}$  such that the diagram

$$\begin{array}{ccc}
 \bar{S} & \xrightarrow{\bar{\gamma}} & \bar{S}/x_1\bar{S} \\
 \uparrow i & & \uparrow \beta \\
 S & \xrightarrow{\gamma} & S/x_1S
 \end{array}$$

is commutative where  $\gamma$  and  $\bar{\gamma}$  are natural epimorphisms and  $i$  is the natural injection. By Lemma 1,  $(x_1S) \cap R = M$  and  $(x_1\bar{S} \cap \bar{R}) = \bar{M}$ , hence we have natural monomorphisms  $\mu: \bar{R}/\bar{M} \rightarrow \bar{S}/x_1\bar{S}$  and  $\mu: R/M \rightarrow S/x_1S$ . It is obvious that the diagram

$$\begin{array}{ccc}
 \bar{R}/\bar{M} & \xrightarrow{\mu} & \bar{S}/x_1\bar{S} \\
 \uparrow \alpha & & \uparrow \beta \\
 R/M & \xrightarrow{\mu} & S/x_1S
 \end{array}$$

is commutative. Since  $\alpha$  is an epimorphism and since  $\bar{S}/x_1\bar{S}$  is generated over  $\mu(\bar{R}/\bar{M})$  by  $\mu(x_2/x_1), \dots, \mu(x_n/x_1)$ , where  $x_2/x_1, \dots, x_n/x_1$  are in  $S$ , we can conclude that  $\beta$  is an epimorphism. Therefore, via  $\bar{\gamma}^{-1}$  and  $\gamma^{-1}\beta^{-1}$  we conclude that  $i$  maps the set of all maximal ideals in  $S$  containing  $x_1$  in a one to one manner onto the set of all maximal ideals in  $\bar{S}$  containing  $x_1$ ; and that if  $\bar{P}$  and  $P$  are corresponding members, then  $P\bar{S} = \bar{P}$ , and the natural monomorphism  $S/P \rightarrow \bar{S}/\bar{P}$  is an epimorphism; hence  $\bar{S}_{\bar{P}}$  has center  $PS_P$  in  $S_P$ .  $(PS_P)\bar{S}_{\bar{P}} = \bar{P}\bar{S}_{\bar{P}}$ , and the natural monomorphism  $S_P/PS_P \rightarrow \bar{S}_{\bar{P}}/\bar{P}\bar{S}_{\bar{P}}$  is an epimorphism. Now everything follows by applying Lemmas 9, 10, 11.

**PROPOSITION 1.** Let  $(\bar{R}, \bar{M})$  be an  $n$ -dimensional regular domain such that the quotient field  $\bar{K}$  of  $\bar{R}$  contains  $K$  as a subfield,  $\bar{R}$  has center  $\bar{M}$  in  $\bar{R}$ .  $M\bar{R} = \bar{M}$ , and the natural monomorphism  $R/M \rightarrow \bar{R}/\bar{M}$  is an epimorphism.\* For any quadratic transform  $(\bar{R}^*, \bar{M}^*)$  of  $\bar{R}$  set  $\tau\bar{R}^* = \bar{R}^* \cap K$  and  $\tau\bar{M}^* = \bar{M}^* \cap K$ . If  $\bar{R}^*$  is an  $m$ -th quadratic transform of  $\bar{R}$  then  $\tau\bar{R}^*$  is an  $m$ -th quadratic transform of  $R$  and we have:  $\bar{R}^*$  has center  $\tau\bar{M}^*$  in  $\tau\bar{R}^*$ ;  $(\tau\bar{M}^*)^q \bar{R} = \bar{M}^{*q}$  and  $\bar{M}^{*q} \cap (\tau\bar{R}^*) = (\tau\bar{M}^*)^q$  for any non-negative integer  $q$ ;  $A\bar{R}^* \cap (\tau\bar{R}^*) = A$  for any ideal  $A$  in  $\tau\bar{R}^*$ , and  $A$  is principal if and only if  $A\bar{R}^*$  is principal; the natural monomorphism  $\tau\bar{R}^*/\tau\bar{M}^* \rightarrow \bar{R}^*/\bar{M}^*$  is an epimorphism; any completion of  $\bar{R}^*$  is also a completion of  $\tau\bar{R}^*$ ; denoting by  $\bar{R} = \bar{R}_0, \bar{R}_1, \dots, \bar{R}_m = \bar{R}^*$  and  $R = R_0, R_1, \dots, R_m = \tau\bar{R}^*$  the unique sequences

\* These conditions are satisfied if  $(\bar{R}, \bar{M})$  is a completion of  $R$ .

of successive immediate quadratic transforms given by Lemma 11 we have  $\tau \bar{R}_i = R_i$  for  $i = 0, 1, \dots, m$ , and  $R_0, R_1, \dots, R_m$  are the only quadratic transforms of  $R$  which are contained in  $\bar{R}^*$ . Furthermore,  $\tau$  maps the set of all transforms of  $\bar{R}$  in a one to one manner onto the set of all quadratic transforms of  $R$ .

*Proof.* Follows from Lemmas 2, 11, 12.

**4. Additional considerations for quadratic transforms of two dimensional regular local domains.** Throughout this section  $(R, M)$  will denote a two dimensional regular domain with quotient field  $K$ ,  $(\bar{R}, \bar{M})$  will denote a completion of  $R$ , and  $\bar{K}$  will denote the quotient field of  $\bar{R}$ . Note that by [Abhyankar 4, Theorem 3], every two dimensional regular local domain with quotient field  $K$  having center  $M$  in  $R$  is a quadratic transform of  $R$ . We shall, all through the paper use this result tacitly. The use of this result can be avoided by replacing the phrase "two dimensional regular local domain with quotient field  $K$  having center  $M$  in  $R$ " by the phrase "quadratic transform of  $R$ ." However the use of the first phrase will make our definitions and results sound more intrinsic.

**DEFINITION 3.** Let  $A$  be a non-zero principal ideal in  $R$ , and let  $(S, N)$  be a two dimensional regular local domain with quotient field  $K$  having center  $M$  in  $R$ . Then  $AS = Q_1 \cap \dots \cap Q_s \cap Q_{s+1} \cap \dots \cap Q_t$  where  $Q_1, \dots, Q_t$  are uniquely determined primary ideals whose associate prime ideals  $P_1, \dots, P_t$  are distinct one dimensional prime ideals in  $S$ . Since  $M \neq 0$ , there are at most a finite number of one dimensional prime ideals in  $S$  which contain  $M$ . Label the  $P_i$  so that  $M \subsetneq P_i$  for  $i = 1, \dots, s$  and  $M \subset P_i$  for  $i = s + 1, \dots, t$ . We define

$$S^R[A] = \text{the } S^R\text{-transform of } A = Q_1 \cap \dots \cap Q_s$$

$$\mu_{S,R}(A) = \text{the multiplicity of } A \text{ at } S^R = \lambda_S(S^R[A]).$$

Note that  $\text{Rad}_S(S^R[A]) = \text{Rad}_S(S^R[\text{Rad}_R A])$  and  $\mu_{R,R}(A) = \lambda_R(A)$ , and that if  $A = A_1 \dots A_q$  where  $A_1, \dots, A_q$  are principal ideals in  $R$  then  $S^R[A] = S^R[A_1] \dots S^R[A_q]$ .

**LEMMA 13.** Let  $(x, y)$  be a basis of  $M$ . There exists a unique immediate quadratic transform  $T_x$  of  $R$  which does not contain  $y/x$ . Let  $H$  be the set of all immediate quadratic transforms of  $R$  other than  $T_x$ . Let  $S = R[y/x]$  and let  $\tau$  be the canonical homomorphism of  $S$  onto  $S/xS$ , let  $k = \tau(R)$  and  $\bar{y} = \tau(y/x)$ . Then  $k$  is a field,  $\bar{y}$  is transcendental over  $k$  and  $S/xS = k[\bar{y}]$ .

Let  $\bar{W}$  be the set of all maximal ideals in  $[k\bar{y}]$ , and let  $W$  be the set of all maximal ideals in  $S$  containing  $x$ . For  $w$  in  $W$  denote  $\tau(w)$  by  $\bar{w}$ . Then  $w \rightarrow \bar{w}$  is a one to one map of  $W$  onto  $\bar{W}$ , and  $w \rightarrow S_w$  is a one to one map of  $W$  onto  $H$ . For each  $w$  in  $W$  fix a generator  $w'(\bar{y})$  of  $\bar{w}$  and let  $w^*(X, Y)$  be the unique form in  $k[X, Y]$  which is not divisible by  $X$  such that  $w^*(1, \bar{y}) = w'(\bar{y})$ ; let us denote  $S_w$  by  $T_{w^*(X, Y)}$ . Then (i) for any  $w$  in  $W$ ,  $w^*(X, Y)$  is irreducible; (ii) for distinct  $w_1$  and  $w_2$  in  $W$ ,  $w_1^*(X, Y)$  and  $w_2^*(X, Y)$  are co-prime; and (iii) any irreducible form of positive degree in  $k[X, Y]$  which is not divisible by  $X$  is a constant multiple of  $w^*(X, Y)$  for some  $w$  in  $W$ . Let  $z$  be a non-zero element in  $R$ , let  $\xi = \lambda_R(z)$ , and let  $f = f(X, Y)$  be a form in  $k[X, Y]$  which either equals  $X$  or equals  $w^*(X, Y)$  for some  $w$  in  $W$ . Then  $MT_f$  is a one-dimensional prime ideal in  $T_f$ , any generator of  $MT_f$  is part of a minimal basis of the maximal ideal in  $T_f$ , there exists a unique principal ideal  $B$  in  $T_f$  such that  $zT_f = B(MT_f)^\xi$ , and then  $B \not\subset MT_f$ . Furthermore  $B = T_f$  if and only if the reduced  $R$ -leading form of  $z$  with respect to  $(x, y)$  is not divisible by  $f(X, Y)$  in  $(R/M)[X, Y]$ .

*Proof.* By Lemma 1,  $N' = (x/y, y)S'$  is a maximal ideal containing  $y$  in  $S' = R'[x/y]$ , and  $x/y, y$  is a minimal basis of the maximal ideal  $N_X$  in the immediate quadratic transform  $T_X = S'_{N'}$  of  $R$ . Now  $x/y \in N_X$  implies that  $y/x \notin T_X$  and hence  $S \not\subset T_X$ . Next, if  $(T, N)$  is any immediate quadratic transform of  $R$  such that  $y/x \notin T$ , then by Lemma 9,  $x/y \in T$  and, in view of the fact that  $y/x \notin T$ , we have that  $x/y \in N$ , and hence  $N \cap S' \supset N'$ ; since  $N'$  is maximal, Lemma 9 tells us that  $T = T_X$ . Therefore by Lemma 9,  $w \rightarrow S_w$  is a one to one map of  $W$  onto  $H$ . Via  $\tau$ , we also conclude that as  $w$  runs over  $W$ ,  $w^*(X, Y)$  runs over mutually co-prime irreducible forms of positive degree in  $k[X, Y]$  which are prime to  $X$ , and that any such form is a constant multiple of  $w^*(X, Y)$  for some  $w$  in  $W$ .

It follows from Lemma 1 that  $k = \tau(R)$  is a field isomorphic to  $R/M$ , and  $S/xS$  is a polynomial ring in one variable  $\bar{y}$  over  $k$ . Next,  $\lambda_R(z) = \xi$  implies that  $z = \phi(x, y)$  where  $\phi(X, Y)$  is a form of degree  $\xi$  in  $X, Y$  with coefficients in  $R$  but not all in  $M$ , and hence denoting by  $\bar{\phi}(X, Y)$  the form obtained from  $\phi(X, Y)$  by reducing its coefficients modulo  $M$ , we have  $\bar{\phi}(X, Y) \neq 0$ . Now  $z = \phi(x, y) = x^\xi \phi(1, y/x)$ , and  $\phi(1, y/x) \notin xS$  because  $\bar{\phi}(1, Y) \neq 0$  and because  $\bar{y} = \tau(y/x)$  is transcendental over  $k$ . Fix  $w$  in  $W$ , let  $f = f(X, Y) = w^*(X, Y)$  and let  $N_f$  be the maximal ideal in  $T_f$ . Since  $xS$  is a prime ideal contained in  $w$ , we have  $(xT_f) \cap S = xS$ , and hence  $\phi(1, y/x) \notin xT_f$ . By Lemma 1,  $MT_f = xT_f$  and  $x$  is part of a minimal basis of  $N_f$ ; hence  $MT_f$  is a one-dimensional prime ideal in  $T_f$ , and there exists a unique

principal ideal  $B$  in  $T_f$  such that  $zT_f = B(MT_f)^t$ , namely  $B = \phi(1, y/x)T_f$  and  $B \not\subset MT_f$ . Since  $N_f \cap S = w$ ,  $\phi(1, y/x)$  is a unit in  $T_f$  if and only if  $\phi(1, y/x) \notin w$ . Also, via  $\tau$ , we know that  $\phi(1, y/x) \notin w$  if and only if  $w^*(1, \bar{y})$  does not divide  $\bar{\phi}(1, \bar{y})$  in  $k[\bar{y}]$ , i.e., if and only if  $w^*(X, Y)$  does not divide  $\bar{\phi}(X, Y)$ , and it is clear that  $\bar{\phi}(X, Y)$  is the reduced  $R$ -leading form of  $z$ . The case of  $T_X$  is entirely similar.

**LEMMA 14.** *Let  $v$  be a non-real valuation of  $K$  having center  $M$  in  $R$ . Then (i)  $v$  is composed<sup>5</sup> with a unique real discrete valuation  $w$  of  $K$  so that  $v = w \circ v^*$  where  $v^*$  is a valuation of  $R_w/M_w$ ; furthermore,  $v^*$  is also real discrete and  $R_w \supset R$ , and denoting  $M_w \cap R$  by  $P_w$  we have that  $P_w$  either equals  $M$  or is a one dimensional prime ideal in  $R$ . Now assume that  $P_w \neq M$ ; then (ii)  $R_w = R_{P_w}$ ; (iii)  $v$  has a unique extension  $\bar{v}$  to  $\bar{K}$  for which  $R_{\bar{v}} \supset \bar{R}$ ; (iv)  $\bar{v}$  has center  $\bar{M}$  in  $\bar{R}$ ; (v)  $\bar{v} = \bar{w} \circ \bar{v}^*$  where  $\bar{w}$  is a unique real discrete valuation of  $\bar{K}$  and  $\bar{v}^*$  is a unique real discrete valuation of  $R_{\bar{w}}/M_{\bar{w}}$ ; (vi)  $P_{\bar{w}} = \bar{R} \cap M_{\bar{w}}$  is a one-dimensional prime ideal in  $\bar{R}$ ; (vii)  $P_{\bar{w}} \cap R = P_w$ ; and (viii)  $P_w \bar{R} = P_{\bar{w}} H$  where  $H$  is a principal ideal in  $\bar{R}$ . Now let  $P_w$  denote a one-dimensional prime ideal in  $R$ , and let  $w$  be the real discrete valuation of  $K$  with  $R_w = R_{P_w}$ . Then (ix) there is at least one, and at most a finite number of valuations  $v_1^*, \dots, v_s^*$  of  $R_w/M_w$  whose valuation rings contain  $R/P_w$ ; furthermore, (x) each  $v_i^*$  is real discrete and has center  $M/P_w$  in  $R/P_w$ ; (xi)  $v_1 = w \circ v_1^*, \dots, v_s = w \circ v_s^*$  are exactly all the distinct non-real valuations of  $K$  composed with  $w$  and having center  $M$  in  $R$ . (xii) If  $R$  is complete then  $s = 1$ ; hence in the general case we may denote by  $\bar{v}_i$  the unique non-real valuation of  $\bar{K}$  having center  $\bar{M}$  in  $\bar{R}$  which is composed with  $\bar{w}_i$ , where  $P_1, \dots, P_t$  are the distinct one dimensional prime divisors of  $P_w \bar{R}$  in  $\bar{R}$ , and  $R_{\bar{w}_i} = \bar{R}_{P_i}$ ; then (xiii)  $s = t$ , and (xiv) we can label the  $\bar{v}_j$  so that for  $j = 1, \dots, s$ ;  $\bar{v}_j$  is the unique extension of  $v_j$  to  $\bar{K}$  having center  $\bar{M}$  in  $\bar{R}$ .*

*Proof.* (i, ii, ix, x, xi, xii) follow from [Abhyankar 4, Theorem 1; Abhyankar 5, Lemma 4.2 and its proof in Section 15] and Lemma 4 above. By [Abhyankar 2, Lemma 13 of Section 7]  $v$  can be extended to a valuation  $\bar{v}$  of  $\bar{K}$  having center  $\bar{M}$  in  $\bar{R}$ , also any other  $\bar{K}$ -extension of  $v$  whose valuation ring contains  $\bar{R}$  must have center  $\bar{M}$  in  $\bar{R}$  since  $\bar{M}$  is the only ideal in  $\bar{R}$  which contracts to  $M$  in  $R$ . By (i),  $\bar{v} = \bar{w} \circ \bar{v}^*$  where  $\bar{w}$  is a unique real discrete valuation of  $\bar{K}$  and  $\bar{v}^*$  is a unique real discrete valuation of  $R_{\bar{w}}/M_{\bar{w}}$ . Now  $R_w \supset \bar{R}$  and  $M_w \cap \bar{R} \neq 0$ ; hence  $M_{\bar{w}} \cap R \neq 0$ ; therefore  $M_{\bar{w}} \cap R \neq 0$ .

<sup>5</sup> For the concept, notation and results on composite valuations see [Abhyankar 5, Section 14].

Since  $M_v$  and  $M_w$  are the only non-zero prime ideals in  $R_v$ , we must have  $M_{\bar{w}} \cap R_v = M_v$  or  $M_w$ . However  $M_{\bar{w}} \cap R_v = M_v$  would imply that  $\bar{w}$  is a  $\bar{K}$ -extension of  $v$  which cannot be the case since  $\bar{w}$  is real while  $v$  is not; hence  $M_{\bar{w}} \cap R_v = M_w$ , and hence  $\bar{w}$  is an extension of  $w$  to  $\bar{K}$ . Now  $(M_{\bar{w}} \cap \bar{R}) \cap R = M_w \cap R$  is one dimensional, and hence  $P_{\bar{w}} = M_{\bar{w}} \cap \bar{R}$  must be one dimensional; this proves (iv, v, vi) and now (vii, viii) follow immediately. Next, let  $p_i$  and  $q_i$  be the  $K$ -restrictions respectively of  $\bar{v}_i$  and  $\bar{w}_i$ . Then  $p_i$  has center  $M$  in  $R$ . Also  $M_{q_i} \cap R = (M_{w_i} \cap \bar{R}) \cap R = P_i \cap R = P_w$ , and hence  $q_i = w \neq p_i$ . Since  $R_w = R_{q_i} \supset R_{p_i}$ ,  $p_i$  is composed with  $w$ , and hence by (i, ii) we conclude that  $p_i = v_j$  for some  $j$ . Thus,  $K$ -restriction gives a mapping of the set  $\bar{v}_1, \dots, \bar{v}_t$  onto the set  $v_1, \dots, v_s$ ; and hence the proof of (xiii, xiv) would be complete if we show that the inverse of this mapping is single valued, i.e., if we prove the uniqueness part of (iii). Assume, if possible, that  $v$  has two distinct  $\bar{K}$ -extensions  $\bar{v}$  and  $\bar{v}'$  having center  $\bar{M}$  in  $\bar{R}$ . By [Abhyankar 4, Theorem 1],  $\bar{v}$  and  $\bar{v}'$  are of  $\bar{R}$ -dimension zero. By [Abhyankar 4, Lemma 12], there exists  $n$  such that the  $n$ -th quadratic transform  $\bar{S}$  of  $\bar{R}$  along  $\bar{v}$  is different from the  $n$ -th quadratic transform  $\bar{S}'$  of  $\bar{R}$  along  $\bar{v}'$ . Let  $S = \bar{S} \cap K$  and  $S' = \bar{S}' \cap K$ . Then by Proposition 1,  $S$  and  $S'$  are distinct  $n$ -th quadratic transforms of  $R$  and clearly both are along the common  $K$ -restriction  $v$  of  $\bar{v}$  and  $\bar{v}'$ . This contradicts Lemma 11.

**DEFINITION 4.** Let  $A$  be a nonzero principal ideal in  $R$  and let  $A = Q_1 \cap \dots \cap Q_t$  be the decomposition of  $A$  into primary ideals such that the associated prime ideals  $P_1, \dots, P_t$  are distinct one dimensional prime ideals. Let  $v$  be a nonreal valuation of  $K$  having center  $M$  in  $R$  such that, for the real discrete valuation  $w$  of  $K$  with which  $v$  is composed,  $M_w \cap R$  is a one-dimensional prime ideal in  $R$ . If  $M_w \cap R \supset A$  then we shall say that  $v$  is a valuation branch of  $A$  at  $R$ ; note that then for a unique  $i$ ,  $M_w \cap R = P_i$  and we shall say that  $Q_i$  is the  $v$  component of  $A$ . By  $\Theta(A, R)$  we shall denote the set of all valuation branches of  $A$  at  $R$ . By Lemma 14 it follows that the valuation branches of  $A$  at  $R$  are in one to one natural correspondence with the one dimensional prime divisors of  $A\bar{R}$  in  $\bar{R}$ , i.e., "the analytic branches of  $A$  at  $R$ ." Also by Lemma 14,  $\Theta(A, R)$  is a finite set and it is empty if and only if  $A = R$ .

**LEMMA 15.** Let  $(S, N)$  be a two dimensional regular local domain with quotient field  $K$  having center  $M$  in  $R$ , and let  $A$  and  $B$  be two nonzero principal ideals in  $R$ . Then we have the following: (i)  $v \in \Theta(S^R[A], S)$ , if and only if,  $v \in \Theta(A, R)$  and  $v$  has center  $N$  in  $S$ ; (ii) If  $A$  is a product of distinct one dimensional prime ideals in  $R$  (i.e., if  $A \nsubseteq u^2R$  for any non-

unit  $u$  in  $R$ ), then  $S^R[A]$  is a product of distinct one dimensional prime ideals in  $S$ ; (iii) If  $A$  is prime one dimensional ideal in  $R$ , then  $S^R[A]$  either equals  $S$  or is a prime one dimensional ideal in  $S$ ; (iv) If  $\mu_{R,R}(A) \leq 1$ , then  $\mu_{S,R}(A) \leq 1$ ; <sup>2</sup> (v) If  $A$  and  $B$  have no common one dimensional prime ideal factors in  $R$ , then  $S^R[A]$  and  $S^R[B]$  do not have any common one dimensional prime ideal factors in  $S$ . Now let  $(S_1, N_1)$  be a two dimensional regular local domain with quotient field  $K$  having center  $N$  in  $S$ ; then (vi)  $S_1^R[S^R[A]] = S_1^R[A]$ .

*Proof.* First take  $v$  in  $\Theta(S^R[A], S)$ , and let  $w$  be the real discrete valuation of  $K$  with which  $v$  is composed; then  $Q = M_w \cap S$  is a one dimensional prime ideal in  $S$  containing  $S^R[A]$ . Now  $R_w \supset S \supset R$  and  $P = M_w \cap R$  is a prime ideal in  $R$ . Since  $K$  is the quotient field of  $R$ ,  $P \neq 0$ . Also, by the definition of  $S^R[A]$ ,  $MS \not\subset Q$  and hence  $P \neq M$ . Therefore,  $P$  is a one dimensional prime ideal in  $R$ , and  $P \supset A$  since  $Q \supset S^R[A] \supset AS$ ; since  $v$  has center  $N$  in  $S$ ,  $v$  must have center  $M$  in  $R$ ; this shows that  $v \in \Theta(A, R)$ . Now take  $v$  in  $\Theta(A, R)$  such that  $v$  has center  $N$  in  $S$ . Then  $M_w \cap R$  is a one dimensional prime ideal in  $R$  containing  $A$  and  $(M_w \cap S) \cap R = M_w \cap R$ , hence  $M_w \cap S$  is a one dimensional prime ideal in  $S$  containing  $AS$  but not containing  $MS$ ; therefore  $v \in \Theta(S^R[A], S)$ . This proves (i). If  $A$  and  $B$  are as in (v),  $v$  and  $v'$  are elements respectively of  $\Theta(S^R[A], S)$  and  $\Theta(S^R[B], S)$ , and  $w$  and  $w'$  are the real discrete valuations of  $K$  with which, respectively,  $v$  and  $v'$  are composed, then as in the proof of (i),  $(M_w \cap S) \cap R$  and  $(M_{w'} \cap S) \cap R$  are one dimensional prime ideals in  $R$  containing  $A$  and  $B$  respectively; hence these contractions to  $R$  are distinct; hence  $M_w \cap S$  and  $M_{w'} \cap S$  are distinct one dimensional prime ideals in  $S$ ; this shows that  $\Theta(S^R[A], S)$  and  $\Theta(S^R[B], S)$  have no elements in common; hence  $S^R[A]$  and  $S^R[B]$  have no common one dimensional prime ideal factor in  $S$ . This proves (v).

Now assume that  $A$  is a prime one dimensional ideal in  $R$  and  $S^R[A] \neq S$ . Then  $\Theta(S^R[A], S)$  contains a valuation  $v$ . Let  $w$  be the real discrete valuation of  $K$  with which  $v$  is composed. By (i),  $v$  is in  $\Theta(A, R)$  and since  $A$  is prime,  $M_w \cap R = A$ ; hence  $w(A) = 1$ . Therefore  $w(AS) = 1$  and hence  $w(S^R[A]) \leq 1$ . However,  $v \in \Theta(S^R[A], S)$  implies that  $w(S^R[A]) > 0$ ; hence  $w(S^R[A]) = 1$ , i.e.,  $M_w \cap S$  is the  $v$ -component of  $S^R[A]$ . Since  $A$  is prime, every valuation in  $\Theta(A, R)$  is composed with  $w$ ; and hence by (i), so is every valuation in  $\Theta(S^R[A], S)$ . Therefore  $S^R[A] = M_w \cap S$ , which implies  $S^R[A]$  is a one dimensional prime ideal in  $S$ . This proves (iii). Now (ii) follows from (iii) and (v) in view of the fact that if  $A = A_1 \cdots A_t$ , where  $A_1, \cdots, A_t$  are principal ideals in  $R$  then  $S^R[A] = S^R[A_1] \cdots S^R[A_t]$ .

Now assume that  $\mu_{R,R}(A) \leq 1$ . If  $\mu_{R,R}(A) = 0$  then  $A = R$  and hence  $S^R[A] = S$  so that  $\mu_{S,R}(A) = 0$ . Now assume that  $\mu_{R,R}(A) = 1$  and let  $x$  be a generator of  $A$ . Then there exists  $y$  in  $R$  such that  $(x, y)$  is a minimal basis of  $M$ . If  $x/y$  is not in  $S$  then by Lemma 9,  $MS = xS$  and hence  $S^R[A] = S$ ; if  $x/y$  is in  $S$  but not in  $N$  then by Lemma 9,  $MS = yS$  and  $AS = y(x/y)S = MS$  and hence  $S^R[A] = S$ ; finally assume that  $x/y$  is in  $N$ , then  $x/y \in R[x/y] \cap N$ , since  $(x/y, y)R[x/y]$  maps onto the prime ideal generated by the  $yR[x/y]$ -residue of  $x/y$  in  $R[x/y]/yR[x/y]$ , we conclude that  $(x/y, y)$  is a prime ideal in  $R[x/y]$  containing  $y$ , and hence  $(x/y, y) = R[x/y] \cap N$ ; this implies that  $(x/y, y)$  is a minimal basis of  $N$ , hence  $(x/y)S$  and  $MS = yS$  do not have a common one dimensional prime divisor in  $S$ , and  $AS = ((x/y)y)S$ ; from this we conclude that  $S^R[A] = (x/y)S$  and  $\lambda_S(x/y) = 1$ , i.e.,  $\mu_{S,R}(A) = 1$ . This proves (iv).

LEMMA 16. Let  $A$  be a nonzero principal ideal in  $R$  other than  $R$ . Let  $v_1, \dots, v_t$  be the distinct members of  $\Theta(A, R)$ . Then  $t \geq 1$  and for each  $i \leq t$  and each  $m \geq 0$  there exists a unique  $m$ -th quadratic transform  $(S_{mi}, N_{mi})$  of  $R$  such that  $v_i$  has center  $N_{mi}$  in  $S_{mi}$ . If  $S_m$  is any  $m$ -th quadratic transform of  $R$ , then  $\mu_{S_m, R}(A) \neq 0$  if and only if  $S_m = S_{mi}$  for some  $i$ .<sup>6</sup>

*Proof.* By [Abhyankar 4, Theorem 1], each  $v_i$  is of  $R$ -dimension zero and now everything follows from Lemma 15 and Definition 2.

DEFINITION 5. Let  $V$  be a nonsingular projective algebraic surface over an algebraically closed ground field  $k$  and let  $P$  be a point on  $V$ . Then there exists a nonsingular projective algebraic surface  $V_1$  and a regular birational map  $f_1$  of  $V_1$  onto  $V$  such that  $P$  is the only fundamental point of  $f_1^{-1}$ ,  $f_1^{-1}(P)$  is a nonsingular irreducible algebraic curve biregularly equivalent to the projective line over  $k$ , and the set of quotient rings of all the points of  $f_1^{-1}(P)$  on  $V_1$  coincides with the set of all immediate quadratic transforms of  $Q(P, V)$ , [See Zariski 14]; we shall say that the pair  $(V_1, f_1)$  is an immediate quadratic transform of  $V$  with center at  $P$ . If  $(V_2, f_2)$  is an immediate quadratic transform of  $V_1$  (for which the center may or may not be in  $f_1^{-1}(P)$ ) then we shall say that  $(V_2, f_1 f_2)$  is a second quadratic transform of  $V$ . In this manner by induction we define an  $m$ -th quadratic transform<sup>7</sup> of  $V$  and finally

<sup>6</sup> We may have  $S_{mi} = S_{mj}$  for  $i \neq j$ . More information concerning this is provided by Lemmas 13 and 14. The consequence of Lemma 16, to the effect that for each  $m$  there is at least one and at most a finite number of  $m$ -th quadratic transforms  $S_m$  of  $R$  for which  $\mu_{S_m, R}(A) \neq 0$ , can also immediately be deduced from Lemma 13.

<sup>7</sup> Note that if  $P^*$  is a point on an  $m$ -th quadratic transform  $(V^*, f)$  of  $V$ , then  $Q(P^*, V^*)$  is an  $q$ -th quadratic transform of  $Q(f(P^*), V)$  for some  $q \leq m$ .

a quadratic transform is to mean an  $m$ -th quadratic transform for some  $m$  (for  $m=0$ ,  $(V, 1)$  is to be the only 0-th quadratic transform of  $V$ ).

Now let  $W$  be a curve on  $V$  with irreducible components  $W_1, \dots, W_t$  and let  $(V^*, f)$  be a quadratic transform of  $V$ , let  $W^* = f^{-1}[W]$ , let  $P^*$  be a point of  $V^*$  and let  $P = f(P^*)$ . Let  $A = M(P, W, V)$  = the ideal of  $W$  at  $P$  on  $V$ , let  $B$  be a defining ideal of  $W$  at  $P$  on  $V$ , and let  $A^* = M(P^*, W^*, V^*)$  be the ideal of  $W^*$  at  $P^*$  on  $V^*$ . Then it is clear that  $f^{-1}[W_1], \dots, f^{-1}[W_t]$  are the distinct irreducible components of  $W^*$ . Let  $R = Q(P, V)$  and  $S = Q(P^*, V^*)$ . We define:

$\lambda(W; P, V)$  = multiplicity of  $W$  at  $P$  on  $V = \lambda_R(A)$

$\mu(W; P^*, V^*, f)$  = multiplicity of  $W$  at  $P^*$  on  $V^*$  for the map  $f = \mu_{S,R}(A)$ .

From the results of this section, we at once deduce the following: (1)  $A^* = S^R[A]$ , (2)  $S^R[B]$  is a defining ideal of  $f^{-1}[W]$  at  $P^*$  on  $V^*$ , (3)  $BS$  is a defining ideal of  $f^{-1}(W)$  at  $P^*$  on  $V^*$ , (4)  $\mu(W; P^*, V^*, f) = \lambda(W^*; P^*, V^*)$ .

LEMMA 17. Assume that  $R$  is complete, let  $z$  be a nonzero nonunit in  $R$ , and let  $\bar{\Lambda}(z) = f_1 \cdots f_t$  be a factorization of  $\bar{\Lambda}(z)$  into pairwise coprime forms  $f_1, \dots, f_t$ . Then  $z = z_1 \cdots z_t$  where  $z_1, \dots, z_t$  are elements of  $R$  with  $\bar{\Lambda}(z_i) = f_i$  for  $i=1, \dots, t$ . In particular, if  $\bar{\Lambda}(z)$  factors into two coprime linear forms, then  $z = z_1 z_2$  where  $(z_1, z_2)$  is a basis of  $M$ .

Proof. Everything will follow if we show that,  $\bar{\Lambda}(z) = fg$  where  $f$  and  $g$  are coprime forms implies that  $z$  is reducible in  $R$ . Assume the contrary. Then by Lemma 14,  $\Theta(zR, R)$  contains a single element  $v$ . Since  $f$  and  $g$  are coprime, by Lemma 13 there exist two distinct immediate quadratic transforms  $S$  and  $S_1$  of  $R$  such that  $S^R[zR] \neq S$  and  $S_1^R[zR] \neq S_1$ , however in view of Lemma 16 this is a contradiction. Hence the lemma is proved.

Now using Lemma 3 we shall give another proof in case  $R$  and  $R/M$  have the same characteristic and either  $R/M$  is infinite or  $\lambda(z) \leq 2$ . Again everything will follow if assuming that  $\bar{\Lambda}(z) = fg$  with coprime forms  $f$  and  $g$  we show that  $z$  is reducible. Let  $k$  be a coefficient field in  $R$ . In view of Lemma 3 we may assume that  $z = x^q + f_1(y)x^{q-1} + \cdots + f_q(y)$  where  $x, y$  is a basis of  $M$ ,  $q = \lambda(z)$ , and  $f_i(y) \in S = k[[y]]$ . Since  $\lambda_R(z) = q$ , we must have  $\lambda_S(f_i(y)) \geq i$ ; hence  $f_i(y) = y^i g_i(y)$  with  $g_i(y) \in S$ . Taking  $R$ -leading forms with respect to  $(k, x, y)$  we have

$$\Lambda(z) = X^q + g_1(0)YX^{q-1} + \cdots + g_q(0)Y^q = A(X, Y)B(X, Y)$$

where  $A$  and  $B$  are nonconstant coprime forms in  $k[X, Y]$  of degrees  $a$  and  $b$

respectively, such that  $a + b = q$ , the coefficient of  $X^a$  in  $A$  is 1, and the coefficient of  $X^b$  in  $B$  is 1. Hence  $X^a + g_1(0)X^{a-1} + \cdots + g_a(0) = A^*(X)B^*(X)$  where  $A^*(X)$  and  $B^*(X)$  are nonconstant monic coprime polynomials in  $k[X]$  of degrees  $a$  and  $b$  respectively. Let  $F(X) = X^a + g_1(Y)X^{a-1} + \cdots + g_a(Y) \in k[[Y]][X]$ . Then  $F(X) \equiv A^*(X)B^*(X) \pmod{Y}$ . Therefore by Hensel's lemma applied to the polynomial  $F(X)$  over the complete local domain  $k[[Y]]$ , we get

$$F(X) = [X^a + G_1(Y)X^{a-1} + \cdots + G_a(Y)][X^b + H_1(Y)X^{b-1} + \cdots + H_b(Y)]$$

with  $a, b > 0$ ,  $a + b = q$ , and  $G_i(Y), H_i(Y) \in k[[Y]]$ . Therefore

$$\begin{aligned} X^a + g_1(Y)YX^{a-1} + \cdots + g_a(Y)Y^a \\ = [X^a + G_1(Y)YX^{a-1} + \cdots + G_a(Y)Y^a] \\ \times [X^b + H_1(Y)YX^{b-1} + \cdots + H_b(Y)Y^b] \end{aligned}$$

and hence

$$z = [x^a + G_1(y)yx^{a-1} + \cdots + G_a(y)y^a][x^b + H_1(y)xb^{b-1} + \cdots + H_b(y)y^b].$$

*Remark 1.* Lemma 17 is false for complete regular local domains of dimension  $n > 2$  as can be seen from the following example: Let  $x_1, \dots, x_n$  be independent variables over a field  $k$  of characteristic  $p$ , and let  $z = x_1^t + x_2^t + x_3^t + x_1x_2x_3 \in S = k[[x_1, \dots, x_n]]$  where  $t > 3$  and  $t$  is prime to  $p$  in case  $p \neq 0$ . Applying the Jacobian Criterion to the hypersurface  $F$  given by  $X_1^t + X_2^t + X_3^t + X_1X_2X_3 = 0$  in the affine  $n$  space over the algebraic closure  $\bar{k}$  of  $k$  with coordinates  $X_1, \dots, X_t$ , one can verify that the singular locus of  $F$  is the  $n-3$  dimensional linear space:  $X_1 = X_2 = X_3 = 0$ ; hence  $F$  is normal everywhere, hence in particular  $F$  is normal at  $X_1 = \cdots = X_n = 0$ , and hence  $S/zS$  is an integral domain, i.e.,  $z$  is irreducible in  $S$ . However  $\Delta(z) = X_1X_2X_3$  has the pairwise coprime factors  $X_1, X_2, X_3$ .

## 5. Normal crossings in a regular local domain.

**DEFINITION 6.** Let  $(R, M)$  be a regular local domain of dimension  $n$ . let  $(\bar{R}, \bar{M})$  be a completion of  $R$ , and let  $Q$  be an ideal in  $R$ . We shall say that  $Q$  has an  $m$ -fold strong normal crossing at  $R$  if there exists a minimal basis  $x_1, \dots, x_n$  of  $M$  such that  $Q = x_1^{u_1} \cdots x_n^{u_n} R$  with  $u_i > 0$ ;  $m$  is then uniquely determined by  $Q$ ; in fact if  $y_1, \dots, y_n$  is any other minimal basis of  $M$  such that  $Q = y_1^{v_1} \cdots y_n^{v_n} R$  with  $v_i > 0$ , then  $m = q$  and after a suitable relabelling of the  $y_i$  we have  $x_i = \delta_i y_i$  and  $u_i = v_i$  for  $i = 1, \dots, m$  where  $\delta_i$

is a unit in  $R$ .<sup>8</sup> If  $Q\bar{R}$  has an  $m$ -fold strong normal crossing at  $\bar{R}$  then we shall say that  $Q$  has an  $m$ -fold normal crossing at  $R$ . It is obvious that if  $Q$  has an  $m$ -fold strong normal crossing at  $R$ , then  $Q$  has an  $m$ -fold normal crossing at  $R$ . If  $Q$  has an  $m$ -fold normal crossing (respectively:  $m$ -fold strong normal crossing) at  $R$  for some  $m$ , ( $m \leq n$ ), then we shall say that  $Q$  has a normal crossing (respectively: strong normal crossing) at  $R$ .

Now let  $A$  be a principal ideal in  $R$ . Then it is clear that  $A$  has an  $m$ -fold strong normal crossing at  $R$  if and only if  $\text{Rad}_R A$  has an  $m$ -fold strong normal crossing at  $R$ , and this is so if and only if there exists a minimal basis  $x_1, \dots, x_n$  of  $M$  such that  $\text{Rad}_R A = x_1 \cdots x_m R$ . Also,  $A$  has an  $m$ -fold normal crossing at  $R$  if and only if  $\text{Rad}_R A$  has an  $m$ -fold normal crossing at  $R$ , which is so if and only if  $\text{Rad}_{\bar{R}}(A\bar{R})$  has an  $m$ -fold strong normal crossing at  $\bar{R}$ , and this is so if and only if there exists a minimal basis  $x_1, \dots, x_n$  of  $\bar{M}$  such that  $\text{Rad}_{\bar{R}}(A\bar{R}) = x_1 \cdots x_m \bar{R}$ . Furthermore, if  $R$  is the quotient ring of a point on an  $n$ -dimensional irreducible algebraic variety, if  $A$  is a defining ideal of a pure  $(n-1)$ -dimensional subvariety  $W$  of  $V$  at  $P$ , and if  $B$  is the ideal of  $W$  at  $P$  on  $V$ , then  $\text{Rad}_R A = B$  and  $\text{Rad}_{\bar{R}}(A\bar{R}) = (\text{Rad}_R A)\bar{R} = B\bar{R}$ . Hence in the geometric case the definitions given here coincide with those of Part I.

LEMMA 18. Let  $(R, M)$  be a regular local domain of dimension  $n > 1$ , let  $Q$  be an ideal in  $R$ , and let  $(R^*, M^*)$  be a quadratic transform of  $R$ . (i) If  $Q$  has a normal crossing (respectively: a strong normal crossing) at  $R$ , then  $QR^*$  has a normal crossing (respectively: a strong normal crossing) at  $R^*$ . (ii) If  $Q$  contains an ideal having a normal crossing (respectively: a strong normal crossing) at  $R$ , then  $QR^*$  contains an ideal having a normal crossing (respectively: a strong normal crossing) at  $R^*$ . (iii) If  $Q$  is a principal ideal in  $R$ , then  $QR^*$  has an  $m$ -fold normal crossing (respectively: an  $m$ -fold strong normal crossing) at  $R$  if and only if  $(\text{Rad}_R Q)R^*$  has an  $m$ -fold normal crossing (respectively: an  $m$ -fold strong normal crossing) at  $R^*$ . (iv) If  $R^* \neq R$ , then  $MR^*$  has an  $h$ -fold strong normal crossing at  $R^*$  with  $h > 0$ .

*Proof.* (iii) follows from the equality:

<sup>8</sup> If  $q = 0$  then  $Q = R$  and hence  $m = 0$ , now assume that  $q > 0$ . Then  $x_1^{u_1} \cdots x_m^{u_m} \in Q \subset y_1 R$  and  $y_1 R$  is prime; hence  $x_i \in y_1 R$  for some  $i \leq m$ . Relabel the  $x_j$  so that  $x_i \in y_1 R$ . Since  $x_i \notin M^2$ , we get  $x_i R = y_1 R$ . Since  $x_1, \dots, x_n$  and  $y_1, \dots, y_n$  are minimal bases of  $M$ , we get  $x_i \notin x_1 R$  and  $y_i \notin y_1 R$  for any  $i > 1$ . Hence  $x_1^{u_1} \cdots x_m^{u_m}$  is in  $x_1^{u_1} R$  but not in  $x_1^{u_1+1} R$ , and  $y_1^{v_1} \cdots y_q^{v_q}$  is in  $y_1^{v_1} R$  but not in  $y_1^{v_1+1} R$ ; i.e.,  $Q$  is contained in  $x_1^{u_1} R$  but not in  $x_1^{u_1+1} R$ , and  $Q$  is contained in  $x_1^{u_1} R$  but not in  $x_1^{u_1+1} R$ . Therefore  $u_1 = v_1$ , hence  $x_1^{u_1} \cdots x_m^{u_m} R = y_1^{v_1} \cdots y_q^{v_q} R$  and we can apply induction.

$$\text{Rad}_{R^*}((\text{Rad}_R Q)R^*) = \text{Rad}_{R^*}(QR^*);$$

(ii) follows from (i); and in view of Lemma 9, (iv) also follows from (i); also in view of Lemma 11 and Proposition 1, it is enough to prove (i) for strong normal crossings in the case when  $R^*$  is an immediate quadratic transform of  $R$ . Let then  $x_1, \dots, x_n$  be a minimal basis of  $M$  such that  $Q = x_1^{u_1} \cdots x_n^{u_n} R$  with  $u_i \geq 0$ . By Lemma 9, we can relabel the  $x_i$  so that  $y_i = x_i/x_1 \in R^*$  for all  $i$ , and we may further relabel the  $y_i$  so that  $y_2, \dots, y_q \in M^*$  and  $y_{q+1}, \dots, y_n \notin M^*$ . Let  $S = R[y_2, \dots, y_n]$ ,  $N = S \cap M^*$ , let  $\tau$  be the natural homomorphism of  $S$  onto  $S/x_1 S$ , let  $\sigma$  be the natural homomorphism of  $S/x_1 S$  onto  $T = S/(x_1, y_2, \dots, y_q)S$ . Since  $S/x_1 S$  is a polynomial ring in the  $n-1$  variables  $\tau y_2, \dots, \tau y_n$ ;  $T$  must be a polynomial ring in the  $n-q$  variables  $\sigma \tau y_{q+1}, \dots, \sigma \tau y_n$ . Also  $x_1 y_2, \dots, y_q \in M^* \cap S = N$  implies that  $\sigma \tau N$  is a maximal ideal in  $T$  and hence has a basis of  $n-q$  elements  $\bar{z}_{q+1}, \dots, \bar{z}_n$ . Fix  $z_i$  in  $S$  such that  $\sigma \tau z_i = \bar{z}_i$  for  $i = q+1, \dots, n$ ; let  $z_1 = x_1$  and  $z_i = y_i$  for  $i = 2, \dots, q$ . Then  $z_1, \dots, z_n$  is a basis of  $N$  and hence a minimal basis of  $M^*$ . Since  $y_{q+1}, \dots, y_n$  are not in  $M^*$ , they are units in  $R^*$ , and we have

$$QR^* = x_1^{u_1} \cdots x_n^{u_n} R^* = z_1^{u_1 + \cdots + u_n} z_2^{u_2} \cdots z_q^{u_q} R^*.$$

LEMMA 19. Let  $(R, M)$  be a two dimensional regular local domain with quotient field  $K$ , let  $A$  be an ideal in  $R$ , let  $(R^*, M^*)$  be a quadratic transform of  $R$  other than  $R$ , and assume that  $A$  has a normal crossing at  $R$  such that either (i)  $A\bar{R} = y_1^{u_1} y_2^{u_2} \bar{R}$  where  $(y_1, y_2)$  is a basis of  $\bar{M}$  and  $u_1 \leq 1$  and  $u_2 \leq 1$ , or (ii) for every one dimensional prime ideal  $H$  in the immediate quadratic transform  $R_1$  of  $R$  contained in  $R^*$  we have that the completion of  $R_1/H$  has no nonzero nilpotent elements.<sup>9</sup> Then  $AR^*$  has a strong normal crossing at  $R^*$ .

*Proof.* Because of Lemma 18, we may assume that  $R^*$  is an immediate quadratic transform of  $R$ . Let  $(\bar{R}^*, \bar{M}^*)$  be the completion of  $R^*$ . Let  $x_1, x_2$  be a minimal basis of  $M$  and let  $y_1, y_2$  be a minimal basis of  $\bar{M}$  such that  $A\bar{R} = y_1^{u_1} y_2^{u_2} \bar{R}$ . Let  $k$  be a representative set of  $R/M$  in  $R$ . Then

\* Note that assumption (ii) implies and is hence equivalent to the following: if a nonzero nonunit  $a$  in  $R_1$  is not divisible by the square of any nonunit in  $R_1$ , i. e., if  $a$  is a product of pairwise coprime nonunits in  $R_1$ , then it is so also in the completion  $\bar{R}_1$  of  $R_1$ . This is clear if  $a$  is irreducible in  $R_1$  and the general case now follows from the fact that if  $b$  and  $c$  are coprime nonunits in  $R_1$  then  $(b, c)R_1$  is primary for the maximal ideal in  $R_1$  and hence  $(b, c)\bar{R}_1$  is primary for the maximal ideal in  $\bar{R}_1$  and therefore  $b$  and  $c$  are coprime in  $\bar{R}_1$ . Also note that assumption (ii) is satisfied if  $R$  is the quotient ring of a point on an algebraic surface.

$y_i = y_i^* + a_{i1}x_1 + a_{i2}x_2$  with  $y_i^* \in \bar{M}^2$  and  $a_{ij} \in k$ . Since  $(x_1, x_2)$  and  $(y_1, y_2)$  are both minimal bases of  $\bar{M}$  we must have  $\det |a_{ij}| \notin \bar{M}^2$ ; hence  $z_1, z_2$  is also a minimal basis of  $\bar{M}$  where  $z_i = y_i - y_i^*$ . Since  $z_i \in R$ ,  $(z_1, z_2)$  is also a minimal basis of  $M$ . Relabel the  $y_i$  so that  $z_2/z_1 \in R^*$ . Let  $w_1 = z_1$ . As in the proof of Lemma 18, either  $z_2/z_1 \in M^*$  in which case, setting  $w_2 = z_2/z_1$ ,  $(w_1, w_2)$  becomes a minimal basis of  $M^*$ ; or  $z_2/z_1$  is a unit in  $R^*$  and hence also in  $\bar{R}^*$ , in which case there exists  $w_2$  in  $R^*$  such that  $(w_1, w_2)$  is a minimal basis of  $M^*$ , and then  $w_2 \neq z_2/z_1$ . Now  $y_i^* = f_i(z_1, z_2)$  where  $f_i(Z_1, Z_2)$  is a form of degree 2 in  $Z_1, Z_2$  with coefficients in  $\bar{R}$ , hence  $y_i^* = f_i(z_1, z_2) = w_1^2 q_i$  with  $q_i = f_i(1, z_2/z_1) \in \bar{R}^*$ . Therefore  $y_1 = z_1 + y_1^* = w_1 + w_1^2 q_1 = w_1 \delta_1$  where  $\delta_1 = 1 + w_1 q_1$  is a unit in  $\bar{R}^*$ ; and  $y_2 = z_2 + y_2^* = w_1(z_2/z_1) + w_1^2 q_2 = w_1 w$  where  $w = (z_2/z_1) + w_1 q_2$ . In case  $z_2/z_1$  is a unit in  $\bar{R}^*$ ,  $w$  is also a unit in  $\bar{R}^*$  and we have  $A\bar{R}^* = w_1^u \bar{R}^*$  with  $u = u_1 + u_2$ , and hence  $AR^* = w_1^u R^*$  because  $w_1 \in R^*$ . In case  $z_2/z_1$  is in  $M^*$ ,  $w_2 = z_2/z_1$ , and hence  $w \equiv w_2 \pmod{w_1}$ , and this implies that  $(w_1, w)$  is also a basis of  $\bar{M}^*$  and  $A\bar{R}^* = w_1^u w^v \bar{R}^*$  where  $v = u_2$ . By Lemma 2, there exists  $t$  in  $AR^*$  such that  $t = w_1^u w^v \delta$  where  $\delta$  is a unit in  $\bar{R}^*$ . Since  $w_1$  is in  $R^*$ ,  $w^v \delta \in \bar{R}^* \cap K$  and hence by Lemma 2,  $w^v \delta \in R^*$ . Now  $w^v \delta R^*$  is primary and hence so is  $w^v \delta \bar{R}^* = (w^v \delta \bar{R}^*) \cap R^*$ ; hence  $w^v \delta = q^h d$  where  $q$  is an irreducible nonunit in  $R^*$ ,  $d$  is a unit in  $R^*$ , and  $h \geq 0$ . If  $v = 0$  then  $AR^* = w_1^u R^*$ , so now assume that  $v \geq 1$ . In case of assumption (i),  $(w_1, q)$  is a basis of  $M^*$  and  $AR^* = w_1^u q R^*$ , so now assume that assumption (ii) holds. Then we must have  $q = w \delta'$  where  $\delta'$  is a unit in  $\bar{R}^*$  and hence  $(w_1, q)$  is a basis of  $\bar{M}^*$  and hence also of  $M^*$  and we have  $AR^* = w_1^u q^v R^*$ .

**Remark 2.** Lemma 19 is false if  $(R, M)$  is a regular local domain of dimension  $n > 2$ , in fact it can then happen that  $A$  has a normal crossing at  $R$  and there exists an infinite sequence  $R = R_0, R_1, R_2, \dots$  of successive immediate quadratic transforms such that  $AR_i$  does not have a strong normal crossing at  $R_i$  for any  $i$ . This can be seen from the following example: Let  $k$  be a field and let  $x_1, \dots, x_n, y$  be  $n+1$  independent variables over  $k$  with  $n > 1$ . Let  $x_{ij} = x_i/y^j$ ,  $B_j = k[x_{1j}, \dots, x_{nj}, y]$ ,  $P_j = (x_{1j}, \dots, x_{nj}, y)B_j$ ,  $R_j = (B_j)_{P_j}$ ,  $M_j = P_j R_j$ . Then  $R_j$  is an immediate quadratic transform of  $R_{j-1}$  and the completion  $(\bar{R}_j, \bar{M}_j)$  of  $R_j$  is the ring of formal power series over  $k$  in the  $n+1$  variables  $x_{1j}, \dots, x_{nj}, y$ . Let  $a = x_1^3 + x_2^3 + x_1 x_2$ . Then by Lemmas 13 and 18,  $aR_j$  has a normal crossing at  $R_j$  for all  $j$ . Also  $a = y^{2j}[(x_{1j}^3 + x_{2j}^3)y^j + x_{1j}x_{2j}]$ ; hence if we show that  $(X^3 + Y^3)Z^j + XY$  is an irreducible element of  $k[X, Y, Z]$ , then it will follow that  $(X_{1j}^3 + X_{2j}^3)Y^j + X_{1j}X_{2j}$  is an irreducible element of  $k[X_{1j}, \dots, X_{nj}, Y]$ , and hence that

$(x_1^3 + x_2^3)y^j + x_1x_2^j$  is an irreducible element of  $R_j$  of  $R_j$ -leading degree 2, and from this it will follow that  $aR_j$  does not have a strong normal crossing at  $R_j$ .

For  $j > 0$ , since  $X^3 + Y^3$  and  $XY$  are coprime polynomials in  $X, Y$ , it is enough to show that  $Z^j + (XY)/(X^3 + Y^3)$  is an irreducible element of  $k(X, Y)[Z]$ ; let  $z$  be a root of this polynomial in some extension  $L^*$  of  $L = k(X, Y)$ , let  $v$  be the valuation of  $L$  given by the irreducible element  $X$  of  $k[X, Y]$ , and let  $v^*$  be an  $L^*$ -extension of  $v$ ; then

$$v^*(z^j) = v^*(XY/(X^3 + Y^3)) = v^*(X),$$

i.e.,  $v^*(z) = v^*(X)/j$ ; hence  $[L(z) : L] = j$  and this proves our assertion. For  $j = 0$  we have the polynomial  $F = X^3 + Y^3 + XY$ . If it were reducible then we would have:  $F = (X^2 + \alpha X + \beta)(X + \gamma)$  with  $\alpha, \beta, \gamma$  in  $k[Y]$ ; since  $F = X^3 \pmod{Y}$ , there exists a nonzero element  $d$  in  $k$  such that either (i)  $\beta = dY^2$  and  $\gamma = d^{-1}Y$ , or (ii)  $\beta = dY$  and  $\gamma = d^{-1}Y^2$ . Now  $\alpha + \gamma = (\text{coefficient of } X^2 \text{ in } F) = 0$ , so that  $\alpha = -\gamma$ , and then  $Y = (\text{coefficient of } X \text{ in } F) = \alpha\gamma + \beta = \beta - \gamma^2$ ; hence in case (i):  $Y = dY^2 - d^{-2}Y^2$  and in case (ii):  $Y = dY - d^{-2}Y^4$ ; both of these are contradictions; therefore  $F$  is irreducible.

**LEMMA 20.** *Let  $(R, M)$  be a two dimensional regular local domain, let  $A = Q_1 \cap \cdots \cap Q_t$  where  $Q_1, \dots, Q_t$  are primary ideals belonging to distinct one dimensional prime ideals in  $R$ . Let  $(S, N)$  be a quadratic transform of  $R$ . Assume that each  $Q_i$  has a strong normal crossing at  $R$  and that  $S^R[A]$  has an  $m$ -fold normal crossing at  $S$  with  $m \leq 1$ . Then  $AS$  has a strong normal crossing at  $S$ .*

*Proof.* Now  $\Theta(A, R)$  is the union of the pairwise disjoint sets  $\Theta(Q_1, R), \dots, \Theta(Q_t, R)$ ; by Lemma 14,  $\Theta(S^R[A], S)$  contains at most one element; and by Lemma 15,  $\Theta(S^R[A], S)$  is contained in  $\Theta(A, R)$ . Therefore after a suitable relabelling of the  $Q_i$  we have that  $S^R[Q_i] = S$  for all  $i > 1$ . Hence  $S^R[A] = S^R[Q_1]$ . Therefore we may assume that  $A$  is primary and then the result follows from Lemma 18.

**PROPOSITION 2.** *Let  $(R, M)$  be a two dimensional regular domain which is the quotient ring of a point on an algebraic or absolute<sup>10</sup> surface, and let  $A$  be a nonzero principal ideal in  $R$ . Then there exists an integer  $n$  such that for any  $m$ -th quadratic transform  $S$  of  $R$  with  $m \geq n$ ,  $AS$  has a strong normal crossing at  $S$  and  $S^R[A]$  has an  $h$ -fold strong normal crossing at  $S$  with  $h \leq 1$ .*

<sup>10</sup> For definition see [Abhyankar 4].

*Proof.* Let  $v_1, \dots, v_t$  be the distinct valuation branches of  $A$  at  $R$ . By [Abhyankar 4, Proposition 5], there exists an integer  $n_i$  such that for the  $n_i$ -th quadratic transform  $(S_i, N_i)$  of  $R$  along  $v_i$  there exists a basis  $x_i, y_i$  of  $N_i$  such that  $(S_i)_{(x_i, y_i)}$  is the valuation ring of the real discrete valuation of the quotient field of  $R$  with which  $v_i$  is composed. Invoking Lemma 12 of [Abhyankar 4], we may assume that  $R_{v_i} \supset S_j$  whenever  $i \neq j$ . Then by Lemma 15,  $S_i^R[A] = x_i^{u_i} S_i$ ,  $u_i \geq 0, 1, \dots, t$ . Hence by part (iv) of Lemma 18, each primary component of  $AS_i$  has a strong normal crossing at  $S_i$ . Let  $q = \max(n_1, \dots, n_t)$ . Then by Lemmas 15 and 18, for any  $q$ -th quadratic transform  $T$  of  $R$ , each primary component of  $AT$  has a strong normal crossing at  $T$  and  $T^R[A]$  is primary. Let  $T_i$  be the  $q$ -th quadratic transform of  $R$  along  $v_i$ . Replacing  $R$  by  $T_i$  and repeating the above argument, in view of Lemma 20 we conclude that there exists an integer  $p$  such that for any  $p$ -th quadratic transform  $T^*$  of any  $q$ -th quadratic transform of  $R$  we have that,  $AT^*$  has a strong normal crossing at  $T^*$  and  $T^{*R}[A]$  is primary. Finally, taking  $n = p + q$  and invoking Lemma 18, we are through.

## 6. Strength of a singularity in a two dimensional regular local domain.

Throughout this section  $(R, M)$  will denote a two dimensional regular local domain with quotient field  $K$ ,  $(\bar{R}, \bar{M})$  will denote the completion of  $R$ ,  $\bar{K}$  will denote the quotient field of  $\bar{R}$ , and  $A$  and  $B$  will denote nonzero principal ideals in  $R$ .

**DEFINITION 7.** Let  $S$  be a two dimensional regular local domain with quotient field  $K$  having center  $M$  in  $R$ . We define:

$$\begin{aligned} v(A, B; S, R) &= \text{strength of singularity of } A \text{ on } B \text{ at } S^R \\ &= \begin{cases} 0 & \text{if } \mu_{S, R}(A) \leq 1 \text{ and } BS \text{ has a} \\ & \text{normal crossing at } S; \\ \frac{1}{2}\mu_{S, R}(A)(\mu_{S, R}(A) + 1) & \text{otherwise.} \end{cases} \end{aligned}$$

**LEMMA 21.** Let  $(\bar{S}, \bar{N})$  be an  $n$ -th quadratic transform of  $\bar{R}$ , let  $S = K \cap \bar{S}$  and  $N = K \cap \bar{N}$ . Then (i)  $(S, N)$  is an  $n$ -th quadratic transform of  $R$ ; (ii)  $(S^R[A])\bar{S} = \bar{S}^R[A\bar{R}]$ , (iii)  $\mu_{S, R}(A) = \mu_{\bar{S}, \bar{R}}(A\bar{R})$ , and (iv)  $v(A, B; S, R) = v(A\bar{R}, B\bar{R}; \bar{S}, \bar{R})$ .

*Proof.* (i) follows from Proposition 1. In the proof of (ii) the case  $n = 0$  and the case  $A = R$  being trivial we may assume that  $n > 0$  and  $A \neq R$ . Then  $MS$  and  $\bar{M}\bar{S}$  are principal ideals and they contain  $AS$  and  $A\bar{S}$  respectively, hence we can write  $AS = DE$  where  $D$  and  $E$  are principal ideals in  $S$  such that  $D$  and  $MS$  are not contained in any one dimensional prime ideal in

$S$  and every one dimensional prime ideal in  $S$  which contains  $E$  also contains  $MS$ . Now we must have that  $D + MS$  either equals  $S$  or is primary for  $N$ ; in the former case we must have  $\bar{S} = (D + MS)\bar{S} = D\bar{S} + \bar{M}\bar{S}$ ; and in the latter case, by Proposition 1,  $(D + MS)\bar{S} = D\bar{S} + \bar{M}\bar{S}$  must be primary for  $\bar{N}$ ; hence in either case, no one dimensional prime ideal in  $\bar{S}$  can contain  $D\bar{S}$  as well as  $\bar{M}\bar{S}$ . Now  $A \subset M$  implies that every one dimensional prime ideal in  $S$  which contains  $MS$  also contains  $AS$  and hence it must contain  $E$ . Therefore  $\text{Rad}_S E = \text{Rad}_S(MS)$ , and hence  $\text{Rad}_{\bar{S}}(E\bar{S}) = \text{Rad}_{\bar{S}}((\text{Rad}_S E)\bar{S}) = \text{Rad}_{\bar{S}}((\text{Rad}_S(MS))\bar{S}) = \text{Rad}_{\bar{S}}(M\bar{S}) = \text{Rad}_{\bar{S}}(\bar{M}\bar{S})$ . Therefore every one dimensional prime ideal in  $\bar{S}$  which contains  $E\bar{S}$  also contains  $\bar{M}\bar{S}$ . Hence  $(S^R[A])\bar{S} = D\bar{S} = \bar{S}^R[A\bar{R}]$ . This proves (ii). Now (iii) follows from (ii) in view of Proposition 1. Finally in view of Proposition 1,  $BS$  has a normal crossing at  $S$  if and only if  $B\bar{S}$  has a normal crossing at  $\bar{S}$  and hence (iv) follows from (iii).

DEFINITION 8. We define:

$$v(A, B; R) = \text{strength of singularity of } A \text{ on } B \text{ at } R = \sum_S v(A, B; S, R)$$

where the sum is taken over all two dimensional regular domains  $S$  with quotient field  $K$  having center  $M$  in  $R$ .

PROPOSITION 3. (analytic invariance of the strength of a singularity).

$$v(A, B; R) = v(A\bar{R}, B\bar{R}; \bar{R}).$$

*Proof.* Follows from Lemma 21 and Proposition 1.

PROPOSITION 4. If  $R$  is the quotient ring of a point on an algebraic or absolute<sup>10</sup> surface, and if  $A$  is not contained in the square of any one dimensional prime ideal in  $R$ , then  $v(A, B; R)$  is finite.

*Proof.* Follows from Proposition 2 and Lemmas 15 and 16.

LEMMA 22. If  $R$  is the quotient ring of a point on an algebraic surface, if  $\text{Rad } A \supset \text{Rad } B$ , and if  $A \neq R$ ; then  $v(A, B; R) = 0$ , if and only if,  $\lambda_R(A) = 1$  and  $B$  has a strong normal crossing at  $R$ .

*Proof.* The 'if' part follows from part (iv) of Lemma 15 and part (i) of Lemma 18. Now assume the  $v(A, B; R) = 0$ . Then  $v(A, B; R, R) = 0$  and hence  $\lambda_R(A) = 1$  and  $B$  has a normal crossing at  $R$ . Let  $x$  be a generator of  $A$ . Now  $B\bar{R} = x_1^u y^v \bar{R}$  where  $(x_1, y)$  is a basis of  $\bar{M}$ . Now  $x\bar{R}$  is prime and  $B\bar{R} \subset x\bar{R}$ ; hence either  $x_1$  or  $y$  equals  $x$  times a unit in  $\bar{R}$ , say it is  $x_1$ . Then  $B\bar{R} = x^u y^v \bar{R}$ . By Lemma 2, there exists  $z$  in  $R$  and a unit  $d$

in  $\bar{R}$  such that  $z = x^u y^v d$ . Let  $t_1 = z/x^u$ . Then  $t_1 \in \bar{R} \cap K = R$ . Since  $t_1 \bar{R}$  is primary, so is  $t_1 R = t_1 \bar{R} \cap R$ , i.e.,  $t_1 = t^w e_1$  where  $t$  is an irreducible non-unit in  $R$  and  $e_1$  is a unit in  $R$ . By a well known result, if  $P$  is a one dimensional prime ideal in  $R$  then  $P\bar{R}$  is not contained in the square of any one dimensional prime ideal in  $\bar{R}$ ; hence we must have  $t = ye$  where  $e$  is a unit in  $\bar{R}$ . Therefore  $(x, t)$  is a basis of  $M$  and  $BR = x^u t^v R$ .

LEMMA 23.  $v(A, B; R) = 0$  if and only if  $v(A, B; R, R) = 0$ .

*Proof.* Follows from Lemmas 15 and 18.

LEMMA 24. There are at most a finite number of immediate quadratic transforms  $S_1, \dots, S_n$  of  $R$  such that  $\mu_{S_i, R}(A) \neq 0$ . If  $S$  is an immediate quadratic transform of  $R$  other than  $S_1, \dots, S_n$  then  $v(S^R[A], BS; S) = 0$ . Also

$$v(A, B; R) = v(A, B; R, R) + \sum_{i=1}^n v(S_i^R[A], BS_i; S_i)$$

*Proof.* Follows from Lemmas 15 and 16.

LEMMA 25.  $v(A, B; R) = v(A, \text{Rad}_R B; R)$ . If  $S$  is a quadratic transform of  $R$  then  $v(A, B; S, R) = v(A, \text{Rad}_R B; S, R)$ .

*Proof.* Follows from part (iii) of Lemma 18.

DEFINITION 9. Let  $V$  be a nonsingular projective algebraic surface over an algebraically closed ground field, let  $W$  and  $W^*$  be curves on  $V$  such that  $W \subset W^*$ , let  $P$  be a point on  $V$  and let  $R = Q(P, V)$ ,  $A = M(P, W, V)$ ,  $B = \text{any defining ideal of } W^* \text{ at } P \text{ on } V$ . Then we define:

$$\begin{aligned} v(W, W^*; P, V) &= \text{strength of singularity of } W \\ &\quad \text{on } W^* \text{ at the point } P \text{ of } V \\ &= v(A, B; R). \end{aligned}$$

By Lemma 25 this does not depend on which defining ideal of  $W^*$  at  $P$  on  $V$  we take for  $B$ . Furthermore we define:

$$\begin{aligned} v(W, W^*; V) &= \text{strength of singularities of } W \text{ on} \\ &\quad W^* \text{ for } V \\ &= \sum_{P \in V} v(W, W^*; P, V). \end{aligned}$$

By Proposition 4, each term in the above summation is finite and only a finite number of terms in the summation are nonzero, for if  $P$  is not a singular point of  $W^*$ , i.e., if the multiplicity of  $W^*$  at  $P$  on  $V$  is at most 1,

then  $\nu(W, W; P, V) = 0$ ; therefore  $\nu(W, W^*; V)$  is finite. Now let  $(V^*, f)$  be a quadratic transform of  $V$ , let  $P^* \in f^{-1}(P)$ , and let  $R^* = Q(P^*, V^*)$ . Then we define

$$\begin{aligned} \nu(W, W^*; P^*, V^*, f) &= \text{strength of singularity of} \\ &\quad W \text{ on } W^* \text{ at the point} \\ &\quad P^* \text{ for the map } f \\ &= \nu(A, B; R^*, R). \end{aligned}$$

Again by Lemma 25 this does not depend on which defining ideal of  $W^*$  at  $P$  on  $V$  we take for  $B$ . Also from the results proved until now it follows that  $\nu(W, W^*; P^*, V^*, f) = \nu(f^{-1}[W], f^{-1}(W^*); P^*, V^*)$ .

LEMMA 26. Let  $V$  be a nonsingular projective algebraic surface over an algebraically closed ground field, let  $W$  and  $W^*$  be curves on  $V$  such that  $W \subset W^*$ , let  $P$  be a point of  $V$ , and let  $(V^*, f)$  be an immediate quadratic transform of  $V$  with center at  $P$ . Then (i)  $\nu(W, W^*; V) = 0$  if and only if  $W$  is nonsingular and  $W^*$  has a strong normal crossing at each point of  $W$ ; (ii)  $\nu(f^{-1}[W], f^{-1}(W^*); V^*) = \nu(W, W^*; V) - \nu(W, W^*; P, V, 1)$  where  $1$  denotes the identity map of  $V$  onto itself.

Proof. (i) follows from Lemma 22 and (ii) follows from Lemma 24.

PROPOSITION 5. Let  $V$  be a nonsingular projective algebraic surface over an algebraically closed ground field  $k$ , let  $W$  be an irreducible curve on  $V$ , and let  $W^*$  be a curve on  $V$  containing  $W$ . Assume that  $\dim |W| > 1 + \nu(W, W^*; V)$ . Then there exists a quadratic transform  $(V^*, f)$  of  $V$  such that  $f^{-1}(W^*)$  has a strong normal crossing at each point  $f^{-1}[W]$ , and  $\dim |f^{-1}[W]| > 1$ .

Proof. We shall apply induction on  $\nu(W, W^*; V)$ . If  $\nu(W, W^*; V) = 0$  then in view of Lemma 26 we may take  $V^* = V$ ; so now assume that  $\nu(W, W^*; V) = n > 0$  and that the Proposition is true whenever  $\nu(W, W^*; V) < n$ . By Lemma 23, for some point  $P$  of  $V$  we must have  $\nu(W, W; P, V, 1) \neq 0$ . Let  $(V', g)$  be an immediate quadratic transform of  $V$  with center at  $P$ . Let  $s$  be the multiplicity of  $W$  at  $P$ , then  $s > 0$ . By Lemma 26,

$$\nu(g^{-1}[W], g^{-1}(W^*); V') = \nu(W, W^*; V) - \frac{1}{2}s(s+1) < n.$$

Let  $L$  and  $L'$  respectively denote the  $k$ -vector spaces of all functions  $u$  in  $k(V) = k(V')$  for which, respectively,  $(u) + W \geq 0$  on  $V$  and  $(u) + g^{-1}[W] \geq 0$  on  $V'$ . Then  $L'$  is a subspace of  $L$ . Let  $H = g^{-1}(P)$ ,  $J = g^{-1}[W]$ , and let  $v$  and  $w$  be the valuations of  $k(V)/k$  having, respectively,  $H$  and  $J$  as

centers on  $V'$ . Let  $(R, M)$  be the quotient ring of  $P$  on  $V$ . Let  $y_1, \dots, y_t$  be arbitrary elements of  $L$  with  $t > \frac{1}{2}s(s+1) = \binom{2+s-1}{2}$ . Fix a generator  $z$  of  $M(P, W, V)$  in  $R$ . Since  $(y_i) + W \geq 0$  on  $V$ , it follows that the polar divisor of  $zy_i$  does not go through  $P$  and hence  $zy_i \in R$ . By Lemma 5, there exist elements  $c_1, \dots, c_t$  in  $k$  which are not all zero such that  $c_1zy_1 + \dots + c_tzy_t \in M^s$ . Now  $z \in M^s$  and  $z \notin M^{s-1}$ , which means that  $v(z) = s$  and  $v(c_1zy_1 + \dots + c_tzy_t) \geq s$ ; hence  $v(c_1y_1 + \dots + c_ty_t) \geq 0$ . Since  $(c_1y_1 + \dots + c_ty_t) + W \geq 0$  on  $V$  and,  $v$  is the only valuation of  $k(V)/k$  whose center on  $V'$  is a curve and whose center on  $V$  is a point, we conclude that  $(c_1y_1 + \dots + c_ty_t) + J \geq 0$  on  $V'$ , i.e.,  $c_1y_1 + \dots + c_ty_t \in L'$ . This shows that  $\dim_k(L'/L) \leq \frac{1}{2}s(s+1)$  and hence  $\dim L' \geq \dim L - \frac{1}{2}s(s+1)$ . Therefore

$$\begin{aligned} \dim |J| &\geq \dim |W| - \frac{1}{2}s(s+1) > 1 + v(W, W; V) - \frac{1}{2}s(s+1) \\ &= 1 + v(J, g^{-1}(W^*); V'). \end{aligned}$$

Therefore by the induction hypothesis there exists a quadratic transform  $(V^*, h)$  of  $V'$  such that  $h^{-1}(g^{-1}(W^*))$  has a strong normal crossing at each point of  $h^{-1}[J]$  and  $\dim |h^{-1}[J]| > 1$ . Let  $f = gh$ . Then  $h^{-1}[J] = f^{-1}[W]$  and  $h^{-1}(g^{-1}(W^*)) = f^{-1}(W^*)$ . This completes the proof.

**7. Computations of the strength of a singularity.** Throughout this section,  $(R, M)$  will denote a two dimensional regular local domain with quotient field  $K$ ,  $(\bar{R}, \bar{M})$  will denote the completion of  $R$ ,  $\bar{K}$  will denote the quotient field of  $\bar{R}$ ,  $A$  and  $C$  will denote nonzero principal ideals in  $R$ ,  $B$  will denote a nonzero principal ideal in  $R$  such that  $\text{Rad } B \subset \text{Rad } A$ ; since  $R$  is a unique factorization domain there then exists a unique principal ideal  $B'$  in  $R$  such that  $\text{Rad } B = (\text{Rad } A)B'$ .

Most of the considerations that follow will be independent of the choice of a particular basis of  $M$ , and also of particular generators of the various principal ideals under consideration; hence when there is no danger of confusion, we shall omit the reference to a particular basis of  $M$ , and to a generator of a principal ideal, thus for instance: by  $\bar{\Lambda}(A)$  we shall mean the reduced  $R$ -leading form of some generator of  $A$ ; by " $\bar{\Lambda}(A)$  and  $\bar{\Lambda}(C)$  are coprime" we shall mean " $\bar{\Lambda}_{(x,y)}(A)$  and  $\bar{\Lambda}_{(x,y)}(C)$  are coprime" for a basis  $(x, y)$  of  $M$ , since this is independent of the particular basis of  $M$  as long as the same basis is used in  $\bar{\Lambda}(A)$  and  $\bar{\Lambda}(C)$ .

**DEFINITION 10.** If  $\Lambda(A)$  and  $\Lambda(C)$  are coprime then we shall say that  $A$  and  $C$  are nontangential at  $R$ ; if  $A$  and  $B'$  are nontangential at  $R$  then

we shall say that  $A$  is nontangential on  $B$  at  $R$ . Note that  $A$  and  $C$  are nontangential at  $R$  if and only if  $A\bar{R}$  and  $C\bar{R}$  are nontangential at  $\bar{R}$ ; and  $A$  is nontangential on  $B$  at  $R$  if and only if  $A\bar{R}$  is nontangential on  $B\bar{R}$  at  $\bar{R}$ . If  $\bar{\Lambda}(A)$  is a product of  $s$  pairwise coprime linear factors then we shall say that  $A$  has an  $s$ -fold ordinary point at  $R$ ; note that in this case  $s$  is the multiplicity of  $A$  at  $R$ . Again,  $A$  has an  $s$ -fold ordinary point at  $R$  if and only if  $A\bar{R}$  has an ordinary  $s$ -fold point at  $\bar{R}$  and by Lemma 17, this is so if and only if  $A\bar{R} = P_1 \cdots P_s$  where  $P_1, \dots, P_s$  are distinct one dimensional prime ideals in  $\bar{R}$  such that  $\bar{\Lambda}(P_1), \dots, \bar{\Lambda}(P_s)$  are pairwise coprime linear forms; hence if  $A$  has an  $s$ -fold ordinary point at  $R$  then  $A$  has a normal crossing at  $R$  if and only if  $s \leq 2$  (and then this normal crossing is  $s$ -fold).

Now assume that  $\lambda(A) = \lambda(C) = 1$  and  $A \neq C$ . Then  $A$  and  $C$  are one dimensional prime ideals,  $A \subset C = \bigcap_{n=1}^{\infty} (C + M^n)$ , and  $C \subset A = \bigcap_{n=1}^{\infty} (A + M^n)$ ; hence there exist unique positive integers  $s$  and  $t$  such that  $A \subset (C + M^s)$ ,  $A \not\subset (C + M^{s+1})$ ,  $C \subset (A + M^t)$ ,  $C \not\subset (A + M^{t+1})$ . We want to show that  $s = t$ . Firstly,  $s = 1$  if and only if  $\bar{\Lambda}(A)$  and  $\bar{\Lambda}(C)$  are coprime, which is so if and only if  $t = 1$ . Now assume that  $s > 1$ , then  $t > 1$ . Let  $a$  and  $c$  be generators of  $A$  and  $C$  respectively. Then  $a = cg + f$  with  $g \in R$  and  $f \in M^s$ . Since  $\lambda(f) = s > 1$  and  $\lambda(a) = \lambda(c) = 1$  we must have  $\lambda(g) = 0$ , i.e.,  $g$  is a unit in  $R$ ; hence  $c = (1/g)a - (f/g)$ ,  $1/g \in R$ , and  $-(f/g) \in M^s$ . Hence  $s \leq t$ , and similarly  $t \leq s$ . Note that  $R/A$  and  $R/C$  are one dimensional regular local domains and  $s = t = \lambda_{R/A}((A + C)/A) = \lambda_{R/C}((A + C)/C)$ .

If  $\lambda(A) = \lambda(C) = 1$ ,  $A \neq C$ , then we shall say that  $A$  and  $C$  have an  $(\lambda_{R/C}((A + C)/A))$ -fold contact at  $R$ ; if  $A$  and  $B'$  have an  $s$ -fold contact at  $R$  then we shall say that  $A$  has an  $s$ -fold contact on  $B$  at  $R$ . Again it is clear that  $A$  and  $C$  have an  $s$ -fold contact at  $R$  if and only if  $A\bar{R}$  and  $C\bar{R}$  have an  $s$ -fold contact at  $\bar{R}$ ; and  $A$  has an  $s$ -fold contact on  $B$  at  $R$  if and only if  $A\bar{R}$  has an  $s$ -fold contact on  $B\bar{R}$  at  $\bar{R}$ .

Finally if: (1)  $s = \lambda(A) > 1$ , (2)  $A\bar{R}$  is prime, and (3) there exists  $z$  in  $\bar{R}$  with  $\lambda(z) = 1$  such that  $A \subset z\bar{R} + \bar{M}^{s+1}$  and  $A \not\subset z\bar{R} + \bar{M}^{s+2}$ , then we shall say that  $A$  has an  $s$ -fold cusp at  $R$ . Note that quite generally for  $u \in R$ ,  $A \subset uR + M^{\lambda(A)+1}$  if and only if  $\bar{\Lambda}(u)$  divides  $\bar{\Lambda}(A)$ . Also, if  $A$  has an  $s$ -fold cusp at  $R$  then by Lemma 17,  $\bar{\Lambda}(A)$  is the  $s$ -th power of a linear form.

Now let  $P$  be a point on a projective algebraic surface  $V$  over an algebraically closed ground field  $k$  and assume that  $A, B, C$  are the ideals of curves  $A^*, B^*, C^*$  on  $V$  at  $P$  respectively. If, respectively: (1)  $A$  and  $C$  are nontangential at  $R$ , (2)  $A$  is nontangential on  $B$  at  $R$ , (3)  $A$  has an  $s$ -fold ordinary point at  $R$ , (4)  $A$  and  $C$  have an  $s$ -fold contact at  $R$ , (5)  $A$  has an

$s$ -fold contact on  $B$  at  $R$ , (6)  $A$  has an  $s$ -fold cusp at  $R$ ; then we shall respectively say that: (1)  $A^*$  and  $C^*$  are nontangential at  $P$  on  $V$ ,  $\dots$ , (6)  $A^*$  has an  $s$ -fold cusp at  $P$  on  $V$ . Note that in all these definitions  $B$  may be replaced by any defining ideal of  $B^*$  at  $P$  on  $V$ ; and in (1),  $C$  may be replaced by any defining ideal of  $C^*$  at  $P$  on  $V$ . Note, also, that if (4) holds then from the usual definition of intersection multiplicity it follows that  $s = i(A^* \cdot C^*, P; V)$ . Now assume that  $V$  is the projective plane and that  $A^*$  has an  $s$ -fold cusp at  $P$  on  $V$ , then there exists a unique line  $L^*$  on  $V$  which is called the tangent line to  $A^*$  at  $P$  such that denoting by  $L$  the ideal of  $L^*$  at  $P$  on  $V$ , we have  $A \subset L + M^{s+1}$ ; if furthermore  $A \not\subset L + M^{s+2}$  then we shall say that  $A^*$  has an ordinary  $s$ -fold cusp at  $P$  on  $V$  (equivalently:  $i(A^* \cdot L^*, P; V) = s + 1$ ); note that what we have called an ordinary 2-fold cusp, is in the classical literature called a simple cusp.

**PROPOSITION 6.** If  $A$  and  $C$  are nontangential at  $R$ , then  $v(A, C; R) = v(A, A; R) - v(A, A; R, R) + v(A, C; R, R)$ . If  $A$  is nontangential on  $B$  at  $R$ , then  $v(A, B; R) = v(A, A; R) - v(A, A; R, R) + v(A, B; R, R)$ . If  $B = B^*D$ , and  $A$  and  $D$  are nontangential at  $R$ , then  $v(A, B; R) = v(A, B^*; R) - v(A, B^*; R, R) + v(A, B; R, R)$ . If  $A = A_1 \cdot \dots \cdot A_t$  where  $A_1, \dots, A_t$  are principal ideals in  $R$  which are pairwise nontangential at  $R$ , then  $v(A, A; R) = \sum_{i=1}^t v(A_i, A_i; R) - \sum_{i=1}^t v(A_i, A_i; R, R) + v(A, A; R, R)$ .

*Proof.* Follows from Lemmas 13 and 24.

**PROPOSITION 7.** (see Figure 1 in Remark 3). If  $A$  has an ordinary  $s$ -fold point at  $R$  then  $v(A, A; R) = v(A, A; R, R) = 0$  or  $\frac{1}{2}s(s+1)$  according as  $s=1$  or  $s > 1$ .

*Proof.* By Proposition 3 we may assume the  $R$  is complete. Then by Lemma 17,  $A = A_1 \cdot \dots \cdot A_t$  where  $A_1, \dots, A_t$  are principal ideals in  $R$  whose reduced  $R$ -leading forms are pairwise coprime linear forms. Let  $S$  be an immediate quadratic transform of  $R$ . If  $S^R[A] = S$  then  $v(A, A; S, R) = 0$ ; if  $S^R[A] \neq S$  then by Lemma 13,  $S^R[A] = S^R[A_i]$  for a unique  $i$  and hence  $\text{Rad}_S(AS) = \text{Rad}_S(A_iS)$ ; now  $A_i$  has a normal crossing at  $R$  and hence by Lemma 18 part (i),  $A_iS$  has a normal crossing at  $S$  and hence by Lemma 18 part (iii),  $AS$  has a normal crossing at  $S$ ; also  $\lambda_R(A_i) = 1$  and hence by Lemma 15 part (iv),  $\mu_{S,R}(A_i) \leq 1$  and hence  $v(A, A; S, R) = 0$ . Thus  $v(A, A; S, R) = 0$  for every immediate quadratic transform of  $R$  and hence by Lemma 24,  $v(A, A; R) = v(A, A; R, R)$ . If  $s > 1$  then  $v(A, A; R, R) = \frac{1}{2}s(s+1)$ , if  $s=1$  then  $A$  has a normal crossing at  $R$  and hence  $v(A, A; R, R) = 0$ .

LEMMA 27. Assume that  $A$  and  $C$  have an  $s$ -fold contact at  $R$ , and that  $R$  is complete and has the same characteristic as  $R/M$ . Then  $M$  has a basis  $(x, y)$  such that the line  $x=0$  is nontangential to  $A$  as well as to  $C$ . Let  $(x, y)$  be any such basis of  $M$  and let  $k$  be a coefficient field in  $R$ . Then there exist unique elements  $a_i, c_i$  in  $k$  such that  $a = y + a_1x + a_2x^2 + \cdots$  and  $c = y + c_1x + c_2x^2 + \cdots$  are generators of  $A$  and  $C$  respectively. Furthermore,  $a_i = c_i$  for  $i = 1, 2, \cdots, s-1$  and  $a_s \neq c_s$ .

*Proof.* Since  $\lambda(AC) = 2$ , everything except the last statement follows from Lemma 3, so now we shall prove the last statement. Now  $s=1$  if and only if  $\Lambda(A) = y + a_1x$  and  $\Lambda(C) = y + c_1x$  are coprime, which is so if and only if  $a_1 \neq c_1$ . So now assume that  $s > 1$  and that  $a_1 = c_1$ . Note that  $R$  is the formal power series ring in the two independent variables  $x$  and  $y$  over the field  $k$ . Now  $a \in cR + M^s$ , and hence we can write

$$y + \sum_{i>0} a_i x^i = (y + \sum_{i>0} c_i x^i) \sum_{0 \leq i+j \leq s} g_{i,j} x^i y^j + \sum_{i+j \geq s} h_{i,j} x^i y^j$$

with  $g_{i,j}$  and  $h_{i,j}$  in  $k$ . Multiplying out the product on the right hand side, we get:

$$\begin{aligned} y + \sum_{i>0} a_i x^i &= \sum_{j=1}^s g_{0,j-1} y^j \\ &\quad + \sum_{i=1}^{s-1} \left( \sum_{p=1}^i c_p g_{i-p,0} + \sum_{j=1}^{s-1-i} (g_{i,j-1} + \sum_{p=1}^i c_p g_{i-p,j}) y^j \right) x^i \\ &\quad + \sum_{i+j \geq s} t_{i,j} x^i y^j \end{aligned}$$

with  $t_{i,j}$  in  $k$ . For  $i=0, 1, \cdots, s-1$  comparing coefficients of  $x^i y^j$  for  $(j=0, 1, \cdots, s-i-1)$  we get the following sets  $I_0, I_1, \cdots, I_{s-1}$  of equations:

$[1 = g_{0,0}, \text{ and for } j=1, \cdots, s-2: 0 = g_{0,j}] \cdots I_0;$   
and

$$[a_i = \sum_{p=1}^i c_p g_{i-p,0}, \text{ and for } j=1, \cdots, s-1-i: g_{i,j-1} = -\sum_{p=1}^i c_p g_{i-p,j}] \cdots I_i$$

for  $i=1, 2, \cdots, s-1$ .

Substituting  $I_0$  in  $I_1$  we get:  $a_1 = c_1$ , and for  $j=1, \cdots, s-2: g_{1,j-1} = -c_1 g_{0,j} = 0$ ; i.e.,

$$[a_1 = c_1, \text{ and for } j=0, \cdots, s-3: g_{1,j} = 0] \cdots J_1.$$

Now substituting  $I_0$  and  $J_1$  in  $I_2$  we get

$$[a_2 = c_2, \text{ and for } j=0, \cdots, s-4: g_{2,j} = 0] \cdots J_2.$$

Now substituting  $(I_0, J_1, J_2)$  in  $I_3$  get  $J_3$ ; and so on. This gives us:  $a_i = c_i$  for  $i = 1, \dots, s-1$ ;  $g_{0,0} = 1$ ; and  $g_{i,j} = 0$  whenever  $0 < i+j < s-2$ . This together with the fact that  $a \notin cR + M^{s+1}$  now tells us that  $a_s \neq c_s$ .

**LEMMA 28.** *If  $A$  and  $C$  have an  $s$ -fold contact at  $R$  with  $s > 1$ , then there exists a unique immediate quadratic transform  $(R_1, M_1)$  of  $R$  for which  $\mu_{R_1, R}(A) \neq 0$ . Furthermore  $R_1^R[A]$  and  $R_1^R[C]$  have an  $s-1$  fold contact at  $R_1$ .*

*Proof.* The first assertion follows from Lemma 13. Now let  $x$  be a generator of  $A$ . Since  $\lambda_R(x) = 1$ , there exists  $y$  in  $R$  such that  $(x, y)$  is a basis of  $M$ . Let  $t = x/y$ ,  $S = R[t]$  and  $N = (x, t)$ . Then by Lemma 13,  $R_1 = S_N$ ,  $M_1 = NR_1 = (x, t)R_1$ , and  $MR_1 = yR_1$ . Let  $A_1 = R_1^R[A]$  and  $C_1 = R_1^R[C]$ . Now  $AR_1 = tyR_1$  and hence  $A_1 = tR_1$ . Let  $z$  be a generator of  $C$ . Because of symmetry in  $A$  and  $C$ , we must have  $z = \tau y$ , where  $\tau$  is in  $R_1$  with  $R_1$ -leading degree 1, and  $C_1 = \tau R_1$ . We can write  $z = f + g + xh$  where  $f \in M^{s+1}$ ,  $g \in M^s$ ,  $g \notin xR$ , and  $h$  is a unit in  $R$ . Now  $f = f^*y^{s+1}$  and  $g = g^*y^s$ , where  $f^*$  and  $g^*$  are in  $S$ ; hence  $\tau = f^*y^s + g^*y^{s-1} + th$ ; therefore  $\tau \in tR_1 + M_1^{s-1}$ . Suppose if possible that  $\tau \in tR_1 + M_1^s$ . Then  $g^*y^{s-1} = t\alpha + \beta$ , where  $\alpha \in R_1$  and  $\beta \in M_1^s$ . Now  $H = R_1/tR_1$  is a one dimensional regular local domain. Let us denote by "bars" the residue classes modulo  $tR_1$ . Then  $\lambda_H(\bar{y}) = 1$ ,  $\lambda_H(\bar{\beta}) \geq s$  and  $\bar{t} = 0$ . Therefore  $\lambda_H(\bar{g}^*) \geq 1$ , i. e.,  $g^* \in tR_1$  and hence  $g^* = \frac{tu}{v}$  where  $u \in S$ ,  $v \in S$ ,  $v \notin N$ . By Lemma 7, we can find  $q > 1$  such that  $uy^q \in M^q$  and  $vy^q \in M^q$ . Also  $v \notin N$  and  $y \in N$  imply that  $v \notin yS$ ; hence by Lemma 7,  $vy^q \notin M^{q+1}$ . Suppose if possible that  $vy^q \in xR$ , then  $vy^q = xw$  with  $w \in M^{q-1}$ ,  $w \notin M^q$ ; hence  $v = (x/y)(w/y^{q-1}) = tw^*$  with  $w^* \in S$ ; therefore  $v \in tS \subset N$ , which is a contradiction. Therefore  $vy^q \notin xR$ . Now we have:  $gv = g^*y^sv = (g^*v)y^s = (tu)y^s = (ty)uy^{s-1} = xuy^{s-1}$ ; hence  $g(vy^q) = x(uy^q)y^{s-1} \in xR$ ; since  $xR$  is prime and  $vy^q \notin xR$  we must have  $g \in xR$ , which is a contradiction. This completes the proof of the lemma.

Now we shall give another proof of the second assertion of Lemma 28 using Lemma 27 in case  $R$  and  $R/M$  have the same characteristic. Without loss of generality we can assume that  $R$  is complete, and hence we may use the notation and result of Lemma 27. Since  $s > 1$ ,  $a_1 = c_1$ ; and replacing  $y$  by  $y + a_1x$  we may assume that  $a_1 = c_1 = 0$ . Let  $t = y/x$ . Then by Lemma 13,  $(x, t)$  is a basis of  $M_1$ . We have

$$\begin{aligned} a &= y + a_2x^2 + a_3x^3 + \dots = tx + a_2x^2 + a_3x^3 + \dots \\ &= x(t + a_2x + a_3x^2 + \dots); \end{aligned}$$

hence

$$R_1^R[A] = (t + a_2x + a_3x^2 + \cdots)R_1,$$

and similarly

$$R_1^R[C] = (t + c_2x + c_3x^2 + \cdots)R_1;$$

and now the assertion follows from Lemma 27.

PROPOSITION 8. (see Figure 2 in Remark 3). *If  $A$  has an  $s$ -fold contact on  $B$ , then  $\nu(A, B; R) = s$  or 0 according as  $s > 1$  or  $s = 1$ .*

*Proof.* If  $s = 1$ , then  $B$  has a normal crossing at  $R$  and  $\lambda(A) = 1$ , and hence  $\nu(A, B; R) = 0$ ; so assume that  $s > 1$ . Now  $\lambda(A) = 1$  and hence by Lemma 13 and Part (v) of Lemma 15, or also by Lemma 16, we can conclude that for each  $n$  there is a unique  $n$ -th quadratic transform  $R_n$  of  $R$  such that  $A_n = R_n^R[A] \neq R_n$ . Now  $R_n$  is an immediate quadratic transform of  $R_{n-1}$  and we have  $\nu(A, B; R) = \sum_{n=0}^{\infty} \nu(A, B; R_n, R)$ . By part (iv) of Lemma 15,  $\nu(A, B; R_n, R) = 0$  or 1 according as  $BR_n$  does or does not have a normal crossing at  $R_n$ . By Lemma 25 we may assume that  $B = AC$  where  $A$  and  $C$  have an  $s$ -fold contact at  $R$ . Let  $C_n = R_n^R[C]$ . In view of part (v) of Lemma 15, Lemma 28 tells us that  $A_n$  and  $C_n$  have an  $s - i$  fold contact at  $R_n$  for  $n = 0, 1, \dots, s - 1$ . Hence for  $n = 0, 1, \dots, s - 2$ ;  $A_n C_n$  and hence  $BR_n$  does not have a normal crossing at  $R_n$ . Also,  $A_{s-1}$  and  $C_{s-1}$  have a 1-fold contact at  $R_{s-1}$ ; hence  $A_{s-1} = xR_{s-1}$  and  $C_{s-1} = yR_{s-1}$  where  $(x, y)$  is a basis of the maximal ideal in  $R_{s-1}$ . Now  $s - 1 > 0$ , and hence  $MR_{s-1} = zR_{s-1}$  where  $z$  is a nonzero nonunit in  $R_{s-1}$ . By part (i) of Lemma 18,  $AR_{s-1}$  and  $CR_{s-1}$  have strong normal crossings at  $R_{s-1}$ , and hence we must have that  $z = u^q d$ , where  $u$  is an element in  $R_{s-1}$  of  $R_{s-1}$ -leading degree 1,  $q > 0$ ,  $d$  is a unit in  $R_{s-1}$ , the reduced  $R_{s-1}$ -leading form of  $z$  is prime to the reduced  $R_{s-1}$ -leading forms of  $x$  and  $y$ , and  $AR_{s-1} = xu^a R_{s-1}$  and  $CR_{s-1} = yu^b R_{s-1}$  with  $a, b > 0$ . Let  $c = a + b$ . Then  $BR_{s-1} = ACR_{s-1} = xyu^c R_{s-1}$ . Therefore,  $BR_{s-1}$  does not have a normal crossing at  $R_{s-1}$ , and  $\text{Rad}_{R_{s-1}}(BR_{s-1})$  has a 3-fold ordinary point at  $R_{s-1}$ ; hence as in the proof of Proposition 7,  $(\text{Rad}_{R_{s-1}}(BR_{s-1}))R_s$  and hence by Lemma 18,  $BR_s$  has a normal crossing at  $R_s$ . Thus  $BR_n$  has a normal crossing at  $R_n$  if and only if  $n \geq s$ ; hence  $\nu(A, B; R) = s$ .

PROPOSITION 9. *If  $A\bar{R} = P_1 P_2$  where  $P_1$  and  $P_2$  are principal ideals in  $\bar{R}$  having and  $s$ -fold contact at  $\bar{R}$ , then  $\nu(A, A; R) = 3s$ .*

*Proof.* Without loss of generality we may assume that  $\bar{R} = R$ . In the

proof of Proposition 8, substitute  $P_1$  for  $A$  and  $P_2$  for  $C$ . Then by Lemma 13 or also by Lemma 15, for  $n=0, 1, \dots, s-1$ ,  $R_n$  is the only  $n$ -th quadratic transform of  $R$  for which  $R_n^R[A] \neq R_n$ ; and for every immediate quadratic transform  $S$  of  $R_{s-1}$ ,  $AS$  has a normal crossing at  $S$ , and  $S^R[A] =$  either  $S^R[P_1]$  or  $S^R[P_2]$ ; and hence by Lemma 15,  $\mu_{S,R}(A) = 1$ . Hence by Lemmas 15 and 18, for every  $m$ -th quadratic transform  $T$  of  $R$  with  $m \geq s$ ,  $AT$  has a normal crossing at  $T$  and  $\mu_{T,R}(A) \leq 1$  and hence  $\nu(A, A; T, R) = 0$ . For  $n=0, 1, \dots, s-1$ ; by Lemma 28,  $R_n^R[P_1]$  and  $R_n^R[P_2]$  have an  $s-n$  fold ordinary point at  $R_n$ , and hence  $\mu_{R_n,R}(A) = \mu_{R_n,R}(P_1) + \mu_{R_n,R}(P_2) = 1 + 1 = 2$ , and therefore  $\nu(A, A; R_n, R) = \frac{1}{2}(2)(2+1) = 3$ . Consequently

$$\nu(A, A; R) = \sum_{n=0}^{s-1} \nu(A, A; R_n, R) = 3s.$$

LEMMA 29. If  $A$  has an  $s$ -fold cusp at  $R$ , then there is a unique immediate quadratic transform  $(R_1, M_1)$  of  $R$  for which  $\mu_{R_1,R}(A) \neq 0$ . Furthermore,  $R_1^R[A]$  has an  $s$ -fold contact on  $AR_1$  at  $R_1$ .

*Proof.* By Proposition 3, we may assume that  $R$  is complete. Let  $z$  be a generator of  $A$ . Then there exists  $x$  in  $R$  with  $\lambda(x) = 1$  such that  $z \in xR + M^{s+1}$  and  $z \notin xR + M^{s+2}$ . We can find  $y$  in  $R$  such that  $(x, y)$  is a basis of  $M$ . If we take leading forms with respect to  $(x, y)$  then  $\bar{\lambda}(z)$  is a nonzero constant multiple of  $\bar{\lambda}(x)^s$  and hence after dividing  $z$  by a suitable unit in  $R$  we can assume that  $z = f + x^s$  where  $f \in M^{s+1}$ . We can write  $f = gx + hy^{s+1}$  where  $g = \sum_{i=0}^s g_i x^i y^{s-i}$  and  $g_i, h \in R$ . Since  $z \notin xR + M^{s+2}$ ,  $h$  must be unit in  $R$ . Let  $t = x/y$ ,  $S = R[t]$ ,  $N = (t, y)S$ . Then by Lemma 13,  $R_1$  is unique, and  $R_1 = S_N$ ,  $M_1 = NR_1 = (t, y)R_1$ , and  $MR_1 = yR$ . Let  $q = h + t \sum_{i=0}^s g_i t^i$ ,  $\tau = t^s + qy$ , and  $A_1 = R_1^R[A]$ . Since  $h$  is a unit in  $R$ , it is also a unit in  $R_1$ . Since  $t \in M_1$  and  $g_i \in R_1$ , we conclude that  $q$  is a unit in  $R_1$ . Now

$$\begin{aligned} z &= x^s + f = x^s + x \left( \sum_{i=0}^s g_i x^i y^{s-i} \right) + hy^{s+1} \\ &= t^s y^s + ty^{s+1} \left( \sum_{i=0}^s g_i t^i \right) + hy^{s+1} \\ &= t^s y^s + qy^{s+1} \\ &= y^s \tau. \end{aligned}$$

Since  $(t, y)$  is a basis of  $M_1$ , we get  $t \notin yM_1$ ; hence  $A_1 = \tau R_1$  and  $\text{Rad}_{R_1}(AR_1) = A_1(MR_1) = y\tau R_1$ . Since  $q$  is a unit in  $R_1$  and  $s > 1$ , it follows that:  $MR_1$  and  $A_1$  have an  $n$ -fold contact at  $R_1$  with  $n \leq s$ . Also, the  $(MR_1)$ -residue

Figure 1.

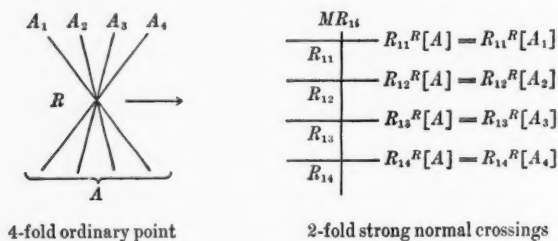


Figure 2.

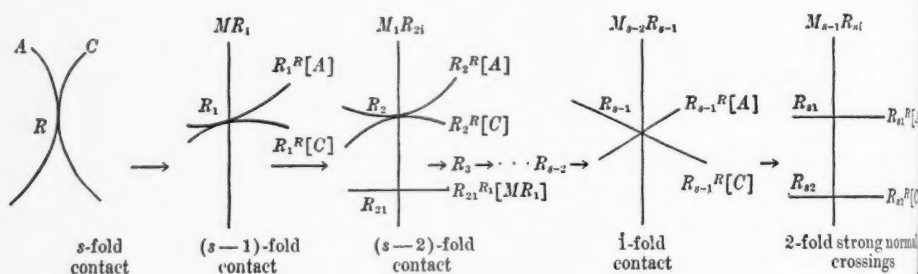


Figure 3.

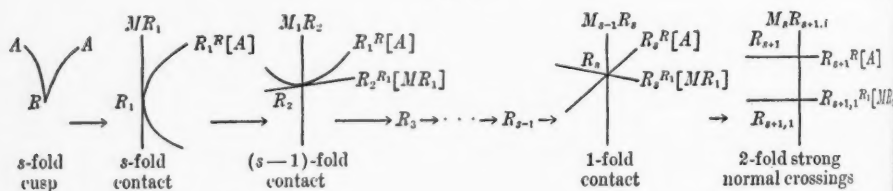
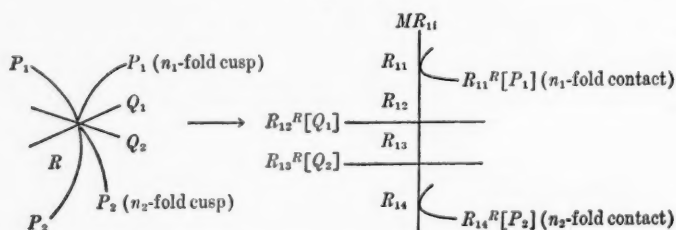


Figure 4.



class of  $t$  generates the maximal ideal in the one dimensional regular local domain  $R_1/MR_1$ , and hence the  $(R_1/MR_1)$ -leading degree of the  $(MR_1)$ -residue class of  $\tau$  equals  $s$ . Therefore  $n = s$ .

**PROPOSITION 10.** (see Figure 3 in Remark 3). *If  $A$  has an  $s$ -fold cusp at  $R$ , then  $v(A, A; R) = \frac{1}{2}s(s+3)$ .*

*Proof.* Let  $R_1$  be the unique immediate quadratic transform of  $R$  for which  $\mu_{R_1, R}(A) \neq 0$ . Then by Lemma 24:

$$v(A, A; R) = v(A, A; R, R) + v(R_1^R[A], AR_1; R_1).$$

Now  $v(A, A; R, R) = \frac{1}{2}s(s+1)$ ; and by Proposition 8 and Lemma 29,  $v(R_1^R[A], AR_1; R_1) = s$ ; hence  $\mu(A, A; R) = \frac{1}{2}s(s+1) + s = \frac{1}{2}s(s+3)$ .

**Remark 3.** It is clear that we can combine Propositions 6 to 10 and their proofs in several ways. For instance, Propositions 6 to 10 tell us that if  $A = P_1 \cdots P_p Q_1 \cdots Q_q$  where  $P_1, \dots, P_p, Q_1, \dots, Q_q$  are pairwise non-tangential principal ideals in  $R$  such that  $P_1, \dots, P_p$  have  $n_1, \dots, n_p$  fold cusps at  $R$ , respectively, and  $Q_1, \dots, Q_q$  have a simple point at  $R$  (i.e.,  $\lambda(Q_i) = 1$ ), and  $\lambda(A) = n_1 + \dots + n_p + q > 1$ , then

$$v(A, A; R) = \frac{1}{2}(n+q)(n+q+1) + n$$

where  $n = n_1 + \dots + n_p$  (see Figure 4 below).

The proofs of Propositions 7, 8, 10, and the above formula, can be illustrated respectively by Figures 1 to 4 on p. 158. In these figures  $(R_{h0}, M_{h0}) = (R_h, M_h)$ ,  $(R_{h1}, M_{h1})$ ,  $(R_{h2}, M_{h2})$ ,  $\dots$  denote distinct immediate quadratic transforms of  $(R_{h-1}, M_{h-1})$ ; also,  $(R_0, M_0)$  is the same as  $(R, M)$ .

**8. Local coverings at a normal crossing.** Throughout this section,  $K$  will denote an  $n$  dimensional algebraic function field over an algebraically closed ground field  $k$  of characteristic  $p$ ,  $K^*$  will denote a galois extension of  $K$ ,  $V$  will denote a normal projective model of  $K/k$ ,  $V^*$  will denote a  $K^*$ -normalization of  $V$ ,  $\phi$  will denote the rational map of  $V^*$  onto  $V$ ,  $P$  will denote a simple point of  $V$  such that  $P$  is tamely ramified in  $K^*$ ,  $P^*$  will denote a point in  $\phi^{-1}(P)$ ,  $(R, M)$  will denote the quotient ring of  $P$  on  $V$ ,  $(R^*, M^*)$  will denote the quotient ring of  $P^*$  on  $V^*$ , and  $W$  will denote a pure  $(n-1)$  dimensional subvariety of  $V$  such that  $\Delta(K^*/V) \subset W$ . Now we state some propositions.

**PROPOSITION 11.** *If  $W$  has a normal crossing at  $P$ , then  $G_1(P^*/P)$  is abelian*

PROPOSITION 12. *If  $W$  has a normal crossing at  $P$  and  $W_1$  is an irreducible component of  $W$  having a simple point at  $P$ , then  $W_1$  does not split locally at  $P^*$ , i. e., only one irreducible component of  $\phi^{-1}(W_1)$  passes through  $P^*$ .*

PROPOSITION 13.  $\Delta(K^*/V)$  is pure  $n-1$  dimensional at  $P$ .

PROPOSITION 11 $\alpha$ . *Proposition 11 under the assumption that  $W$  has a strong normal crossing at  $P$ .*

PROPOSITION 12 $\alpha$ . *Proposition 12 under the assumption that  $W$  has a strong normal crossing at  $P$ .<sup>11</sup>*

PROPOSITION 11 $\beta$ . *Proposition 11 for  $n=2$ .*

PROPOSITION 12 $\beta$ . *Proposition 12 for  $n=2$  (or equivalently Proposition 12 $\alpha$  for  $n=2$ ).*

PROPOSITION 13 $\beta$ . *Proposition 13 for  $n=2$ .*

LEMMA 30. *Proposition 12 under the assumption that  $G_i(P^*/P)$  is abelian.*

*Proof.* Let  $K_i$  be the fixed field of  $G_i(P^*/P)$ , let  $V_i$  be a  $K_i$ -normalization of  $V$ , let  $f$  be the rational map of  $V^*$  onto  $V_i$ , let  $\phi_i$  be the rational map of  $V_i$  onto  $V$ , let  $P_i = f(P^*)$ , and let  $(R_i, M_i)$  be the quotient ring of  $P_i$  on  $V_i$ . Let  $(\bar{R}^*, \bar{M}^*)$ ,  $(\bar{R}_i, \bar{M}_i)$  and  $(\bar{R}, \bar{M})$  be the completions of  $R^*$ ,  $R_i$  and  $R$  respectively. Then we can find a basis  $x_1, \dots, x_n$  of  $\bar{M}$  with  $x_1$  in  $\bar{R}$  such that  $x_1 \cdots x_m \bar{R} = (M(W, P, V)) \bar{R}$  and  $x_1 \bar{R} = M(W_1, P, V)$ . Now  $x_1, \dots, x_n$  is also a basis of  $\bar{M}_i$ ,  $x_1 \in M_i$ ,  $x_1 \notin M_i^2$ ; hence  $\phi_i^{-1}(W)$  also has a normal crossing at  $R_i$ ,

$$x_1 \cdots x_m \bar{R}_i = (M(\phi_i^{-1}(W), P_i, V_i)) \bar{R}_i, \quad x_1 \bar{R}_i = M(\phi_i^{-1}(W_1), P_i, V_i),$$

and  $\phi_i^{-1}(W_1)$  has a simple point at  $P_i$ . Fix generators  $q_1, \dots, q_t$  of the ideals at  $P_i$  on  $V_i$  of the different irreducible components of  $\phi^{-1}(W)$  passing through  $P_i$ . Then each  $q_m$  equals a product of a certain number of the  $x_j$ 's times a unit in  $\bar{R}_i$ . Since  $G(K^*/K_i) = G_i(R^*/R)$  is abelian and its order is prime to  $p$  in case  $p \neq 0$ , and since  $k$  contains all the roots of unity, we can find  $z_1, \dots, z_s$  in  $K^*$  such that  $z_a^{n_a} \in K_i$  and  $K^* = K_i(z_1, \dots, z_s)$  where  $n_a$  is a positive integer which is not divisible by  $p$  in case  $p \neq 0$ . Multiplying  $z_a$  by a suitable element in  $R_i$  we can assume that  $z_a^{n_a} \in R_i$ , and since  $R_i$  is a

<sup>11</sup> Now it is unnecessary to say that  $W_1$  has a simple point at  $P$ , and it is enough to say that "let  $W_1$  be an irreducible component of  $W$  passing through  $P$ ."

unique factorization domain, after dividing  $z_a$  by a suitable element in  $R_i$  we can assume that  $z_a^{n_a} = d_a h_{a1}^{u_{a1}} \cdots h_{ab_a}^{u_{ab_a}}$  where  $h_{a1}, \dots, h_{ab_a}$  are pairwise coprime irreducible nonunits in  $R_i$ ,  $d_a$  is a unit in  $R_i$ , and  $u_{a1}, \dots, u_{ab_a}$  are positive integers less than  $n_a$ . Then the real discrete valuation of  $K_i/k$ , whose valuation ring is the quotient ring of  $R_i$  with respect to the prime ideal  $h_{ac}R_i$ , is ramified in  $K^*$ ; hence the irreducible  $(n-1)$ -dimensional subvariety of  $V$ , whose ideal at  $P_i$  on  $V_i$  is  $h_{ac}R_i$ , is ramified in  $K^*$  and hence it must be an irreducible component of  $\phi^{-1}(W)$  passing through  $P_i$ , i.e.,  $h_{ac}$  must equal some  $q_m$  times a unit in  $R_i$ , i.e.,  $h_{ac}$  must equal the product of a certain number of the  $x_j$ 's times a unit in  $\bar{R}_i$ . Let  $E^*$  and  $E_i$  be the quotient fields of  $\bar{R}^*$  and  $\bar{R}_i$  respectively. Then by Proposition 1 of [Abhyankar 2],  $E^* = E_i(z_1, \dots, z_s)$ . Let  $e = n_1 n_2 \cdots n_s$ ; then  $e$  is not divisible by  $p$  in case  $p \neq 0$ . Let  $E' = E_i(x_1^{1/e}, \dots, x_n^{1/e})$ . Since  $k$  is algebraically closed and  $e$  is not divisible by  $p$  in case  $p \neq 0$ , by Hensel's Lemma, every unit in  $E_i$  has all its  $e$ -th roots in  $E'$  and hence we can conclude that  $E' \supset E^*$ . Let  $(\bar{R}', \bar{M}')$  be the integral closure of  $\bar{R}^*$  in  $E'$ . Now  $\bar{R}_i = k[[x_1, \dots, x_n]]$  and hence by Lemma 5 of [Abhyankar 1],  $\bar{R}' = k[[x_1^{1/e}, \dots, x_n^{1/e}]]$ , and  $x_1^{1/e}, \dots, x_n^{1/e}$  is a minimal basis of  $\bar{M}'$ . Hence  $x_1 \bar{R}'$  is a primary ideal in  $\bar{R}'$ , and therefore  $x_1 \bar{R}^*$  is a primary ideal in  $\bar{R}^*$ . Since  $x_1 \bar{R}^* \cap R^* = x_1 R^*$ , we conclude that  $x_1 R^*$  is a primary ideal in  $R^*$ . Now  $x_1 R^*$  is a defining ideal of  $\phi^{-1}(W_1)$  at  $P^*$  on  $V^*$  and consequently only one irreducible component of  $\phi^{-1}(W_1)$  can pass through  $P^*$ .

*Remark 4.* The proof of Proposition 13 given in [Abhyankar 1, Theorem 1] is incorrect, a correct proof is given in [Zariski 16] and hence now Proposition 11 $\alpha$  follows as in [Abhyankar 1, Theorem 2] and therefore now Proposition 12 $\alpha$  follows from Proposition 11 $\alpha$  and Lemma 30. However, here we shall give a direct proof of Proposition 11 $\beta$  (i.e., Proposition 11 for  $n=2$ ) without using Proposition 13, and from Proposition 11 $\beta$  we shall derive Proposition 13 $\beta$  (i.e., Proposition 13 for  $n=2$ ) and this will then, by Lemma 30, give Proposition 12 $\beta$ . Propositions 11 and 12 were not used in Part I (See Remark 6 below) and will neither be used in this paper; we shall prove them elsewhere. Whereas Zariski's proof (of Proposition 13) uses the Jacobian theory, our proof (of Proposition 11 $\beta$  and hence of Proposition 13 $\beta$ ) will, in a sense, be more geometric and will be based on the following: (1) the transform of a simple point under a quadratic transformation is a projective line, (2) a nonabelian tamely ramified covering of a projective line must have at least three branch points, (3) connectedness theorem of Zariski, and (4) the limit of a quadratic sequence of two dimen-

sional regular local domains is a valuation ring. Now for the case of an arbitrary two dimensional regular domain, (1) is still true (even for unequal characteristic), (3) has recently been proved by Chow [7], and (4) was proved by us in [4]. Hence it is expected that the proofs of Propositions 11 $\beta$ , 12 $\beta$ , 13 $\beta$  to be given below can probably be generalized to the case of an arbitrary two dimensional regular local domain.

**LEMMA 31.** *Proposition 13 $\beta$  under the assumption that  $G(K^*/K)$  is abelian and its order is prime to  $p$  in case  $p \neq 0$ .*

*Proof.* If  $P \notin \Delta(K^*/V)$  then there is nothing to prove, so assume that  $P \in \Delta(K^*/V)$ . Since  $K^*/K$  is abelian, it is a compositum of cyclic extensions and hence by Lemma 12 of Part I, there exists a field  $L$  between  $K$  and  $K^*$  such that  $L/K$  is cyclic of order  $n$  (where  $n$  is prime to  $p$  in case  $p \neq 0$ ) such that  $P \in \Delta(L/V)$ . Now  $L = K(z)$  with  $z^n \in R$ . Since  $R$  is a unique factorization domain, we can arrange matters such that  $z^n = dx_1^{u_1} \cdots x_t^{u_t}$  where  $x_1, \dots, x_t$  are pairwise coprime irreducible nonunits in  $R$ ,  $u_i$  are positive integers less than  $n$ , and  $d$  is a unit in  $R$ . If  $t$  were zero then the discriminant of the minimal monic polynomial of  $z$  over  $K$  would be a unit in  $R$  and hence  $P$  would be unramified in  $L$ , hence  $t \neq 0$ . Now it is clear that the irreducible curve on  $V$ , having  $x_1 R$  for its ideal at  $P$ , is ramified in  $L$  and hence it is ramified in  $K^*$ .

**LEMMA 32.** *Let  $S$  be the formal power series ring in  $n$  variables over  $k$ , let  $E$  be the quotient field of  $S$ , and let  $E^*$  be an abelian extension of  $E$  such that  $[E^*: E]$  is prime to  $p$  in case  $p \neq 0$  and such that for any minimal prime ideal  $I$  in  $S$  the real discrete valuation of  $E$  with valuation ring  $S_I$  is unramified in  $E^*$ . Then  $E^* = E$ .*

*Proof.* Since  $E^*/E$  is abelian, it is enough to show that if any field  $E'$  between  $E$  and  $E^*$  is cyclic over  $E$  then  $E' = E$ . We can find  $z$  in  $E'$  such that  $E' = E(z)$  and  $z^u \in S$  where  $u$  is prime to  $p$  in case  $p \neq 0$ . For any minimal prime ideal  $I$  in  $S$  the valuation with valuation ring  $S_I$  is unramified in  $E^*$  and hence it is unramified in  $E'$ . Since  $S$  is a unique factorization domain we can arrange matters so that  $z^u$  is not divisible by the  $u$ -th power of any irreducible nonunit in  $S$ . If  $z^u$  were then divisible by an irreducible nonunit  $h$  in  $S$  then the valuation with valuation ring  $S_{hS}$  would be ramified in  $E'$ , hence  $z^u$  is a unit in  $S$ . Therefore by Hensel's Lemma  $z \in E$ .

**LEMMA 33.** *In the notation of Proposition 11, if  $G_4(P^*/P)$  is meta-abelian then it is abelian.*

*Proof.* Let  $K_i$  be the fixed field of  $G_i(P^*/P)$ , let  $R_i = K \cap R^*$ ,  $M_i = K \cap M^*$ . Let  $L$  be a field between  $K^*$  and  $K_i$  such that  $K^*/K$  and  $L/K_i$  are abelian. Let  $S = L \cap R^*$ ,  $N = L \cap M^*$ . Let  $(\bar{R}^*, \bar{M}^*)$ ,  $(\bar{S}, \bar{N})$ ,  $(\bar{R}_i, \bar{M}_i)$  be completions, respectively, of  $R^*$ ,  $S$ ,  $R_i$ , and let  $E^*$ ,  $F$ ,  $E_i$  be the quotient fields, respectively, of  $\bar{R}^*$ ,  $\bar{S}$ ,  $\bar{R}_i$ . Now  $R^*$  is the only local ring in  $K^*$  lying above  $R_i$  and hence  $K_i$  is the splitting field of  $R^*/R_i$  as well as that of  $S/R_i$ , and  $L$  is the splitting field of  $R^*/S$ . Therefore by Lemma 7 of [A2],  $E^*/F$ ,  $E^*/E_i$ ,  $F/E_i$  are galois and their galois groups are isomorphic to the galois groups, respectively, of  $K^*/L$ ,  $K^*/K_i$ ,  $L/K_i$ . Let  $V_i$  be a  $K_i$ -normalization of  $V$ , let  $P_i$  be the point of  $V_i$  whose quotient ring on  $V_i$  is  $R_i$ , let  $\phi_i$  be the map of  $V_i$  onto  $V$ , let  $W_1, \dots, W_t$  be the irreducible components of  $\phi_i^{-1}(W)$  passing through  $P_i$ , and let  $H_j = M(P_i, W_j, V_i)$ . Now  $\Delta(K^*/V_i) \subset \phi_i^{-1}(W)$ , and as in the proof of Lemma 30 we get  $(H_1 H_2 \cdots H_t) \bar{R}_i = x_1 \cdots x_m \bar{R}_i$  where  $(x_1, \dots, x_n)$  is a basis of  $\bar{M}_i$ ,  $(m \leq n)$ . Let  $T$  be the integral closure of  $R_i$  in  $K^*$  and let  $\bar{T}$  be the integral closure of  $\bar{R}_i$  in  $E^*$ . Let  $D$  be the ideal in  $R_i$  generated by all  $K_i$ -discriminants of all  $K_i$  bases of  $K^*$  which belong to  $R^*$  and let  $\bar{D}$  be the ideal in  $\bar{R}_i$  generated by all  $E_i$ -discriminants of all  $E_i$ -bases of  $E^*$  which belong to  $\bar{R}^*$ . Since  $[E^*: E_i] = [K^*: K_i]$ , we get  $D \subset \bar{D}$ . The only rank one prime ideals in  $R_i$  which can contain  $D$  are amongst  $H_1, \dots, H_t$  [see, Krull 9] and after a suitable relabelling of the  $H_j$  one can assume that  $H_1, \dots, H_b$  are these ideals and that  $(H_1 \cdots H_b) \bar{R}_i = x_1 \cdots x_h \bar{R}_i$ . Hence  $D = Q_1 \cap \cdots \cap Q_b \cap J_1 \cap \cdots \cap J_a$  where  $Q_j$  is primary for  $H_j$  and  $J_j$  is primary for a prime ideal of rank  $> 1$ . Let  $I$  be a rank one prime ideal in  $\bar{R}_i$  such that  $\bar{D} \subset I$ . Let  $J = J_1 \cdots J_a$  and  $Q = Q_1 \cdots Q_b$ . Then  $(J \bar{R}_i)(Q \bar{R}_i) = (JQ) \bar{R}_i \subset D \bar{R}_i \subset \bar{D} \subset I$ . Suppose if possible that  $J \bar{R}_i \subset I$ . Now  $R_i/J$  is a local ring of dimension less than  $n-1$  and hence its completion  $\bar{R}_i/J \bar{R}_i$  is also a local ring of dimension less than  $n-1$  [Chevalley 6, Proposition 2 of Section III]. Now  $\bar{R}_i/I$  is a homomorphic image of  $\bar{R}_i/J \bar{R}_i$  and hence  $\bar{R}_i/I$  is a local ring of dimension less than  $n-1$  [Chevalley 6, Proposition 1 of Section III], this is a contradiction since  $I$  is a minimal ideal in  $\bar{R}_i$  [Cohen 8, Theorem 8]. Therefore  $J \bar{R}_i \not\subset I$  and hence  $Q \bar{R}_i \subset I$ . Now  $Q \bar{R}_i = x_1^{u_1} \cdots x_h^{u_h} \bar{R}_i$  and hence some  $x_j$  belongs to  $I$ , i.e.,  $I = x_j \bar{R}_i$ . Hence the minimal prime ideals in  $\bar{R}_i$  which are ramified in  $E^*$  are amongst  $x_1 \bar{R}_i, \dots, x_n \bar{R}_i$ . Let  $v_i$  be the valuation of  $E_i$  whose valuation ring is the quotient ring of  $\bar{R}_i$  with respect to  $x_i \bar{R}_i$ . Let  $e_i$  be the ramification index over  $v_i$  of any  $E^*$ -extension of  $v_i$ . Let  $e = e_1 \cdots e_n$ . Let  $E^{*'} = E^*(x_1^{1/e}, \dots, x_n^{1/e})$ ,  $E'_i = E_i(x_1^{1/e}, \dots, x_n^{1/e})$ ,  $F' = F(x_1^{1/e}, \dots, x_n^{1/e})$ . Let  $\bar{R}^{*'}_i$ ,  $\bar{R}'_i$ ,  $\bar{S}'$  be the integral closures of  $\bar{R}_i$  in  $E^{*'}$ ,  $E'_i$ ,  $F'$  respectively. Then using the techniques

of the proofs of Lemma 6 of [Abhyankar 1] and Lemma 9 of Part I, we can conclude that: (i) no minimal prime ideal in  $\bar{R}'_i$  is ramified in  $F'$ , (ii) no minimal prime ideal in  $\bar{R}'_i$  is ramified in  $E^*$ , and (iii) no minimal prime ideal in  $\bar{S}'$  is ramified in  $E^*$ . By [Abhyankar 1, Lemma 5],

$$\bar{R}'_i = k[[x_1^{1/e}, \dots, x_n^{1/e}]]$$

and it is clear that  $E^*/F'$  and  $F'/E'_i$  are abelian extensions. Hence in view of Lemma 32, (i) tells us that  $F' = E'_i$  and then (iii) tells us that  $E^* = E'_i$ . Therefore  $E_i \subset E^* \subset E'_i$ . Now  $E'_i/E_i$  is abelian and hence so is  $E^*/E_i$ . Therefore  $G_i(P^*/P)$  is abelian.

**LEMMA 34.** *Let the assumption be as in Proposition 11 $\beta$  and also assume that  $R^*$  is the only local ring in  $K^*$  lying above  $R$ . Assume if possible that  $G_i(P^*/P)$ , i. e.,  $G_i(R^*/R)$ , is nonabelian. Then there exists an immediate quadratic transform  $S$  of  $R$  such that for any local ring  $S^*$  in  $K^*$  lying above  $S$ ,  $G_i(S^*/S)$  is nonabelian.*

*Proof.* Note that since  $k$  is algebraically closed,  $G_i(P^*/P) = G_s(P^*/P)$  and hence  $G_i(P^*/P) = G(K^*/K)$ . Let  $V_1$  be an immediate quadratic transform of  $V$  with center at  $P$ , let  $f$  be the map of  $V_1$  onto  $V$ , let  $V^*_1$  be a  $K^*$ -normalization of  $V_1$ , let  $\phi_1$  be the map  $V^*_1$  onto  $V_1$  and let  $f^*$  be the map of  $V^*_1$  onto  $V^*$ . Let  $L = f^{-1}(P)$  and let  $L^*_1, \dots, L^*_t$  be the irreducible components of  $L^* = \phi_1^{-1}(L)$ . Since  $P^*$  is the only point on  $V^*$  lying above  $P$ , we have  $f^{*-1}(P^*) = L^*$  and hence by the connectedness theorem [Zariski 13],  $L^*$  must be connected. Let  $W_1 = f^{-1}(W)$ . Then it is clear that  $\Delta(V^*_1/V_1) \subset W_1$ . By Lemma 19,  $W_1$  has a strong normal crossing on  $V_1$  at each point of  $L$ . We now have to consider two cases according as  $t = 1$  or  $t > 1$ . First take the case  $t > 1$ . Since  $L^*$  is connected, after a suitable relabelling of the  $L^*_j$ , we can assume that  $L^*_1$  and  $L^*_2$  have a point  $P^*_1$  in common. Let  $\phi_1(P^*_1) = P_1$ . Then by Lemma 30,  $G_i(P^*_1/P_1)$  is nonabelian and we can take  $S = Q(P_1, V_1)$ . Next, take the case  $t = 1$ . Since  $L$  does not split in  $K^*$ , the inertia field  $K'$  of  $L^*_1/L$  is a galois extension of  $K$  and  $L$  is unramified in  $K'$ . Let  $V'_1$  be a  $K'$ -normalization of  $V$  and let  $\phi'$  be the map of  $V'_1$  onto  $V_1$ . Let

$$L' = \phi'^{-1}(L), \text{ and } H = Q(L, V_1)/M(L, V_1) \subset H' = Q(L', V'_1)/M(L', V'_1).$$

Then  $H'/H$  is a galois extension whose galois group is isomorphic to the galois group of  $K'/K$ . Let  $L_H$  be an  $H'$ -normalization of  $L$  and let  $\phi_H$  be the map of  $L_H$  onto  $L$ . Then  $\phi_H$  is the natural "lifting" to  $L_H$  of the  $L'$ -restriction of  $\phi'$ . Hence it is clear that  $\Delta(L_H/L) \subset \Delta(V'_1/V_1) \cap L$ . By

Lemma 14 of Part I,  $G(K^*/K')$  is cyclic and hence by Lemma 33,  $G(K'/K)$  is nonabelian; hence  $G(H'/H)$  is nonabelian. Now  $H/k$  is simple transcendental and  $L$  is nonsingular (i.e.,  $L$  is biregularly equivalent to the projective line over  $k$ ) and hence by [Abhyankar 3, Proposition 6],  $\Delta(L_H/L)$  contains at least three distinct points and hence by Lemma 13, there exists a point  $P_1$  in  $\Delta(V'_1/V) \cap L$  such that  $P_1 \notin f^{-1}[W]$ . Therefore  $P_1$  is an isolated point of  $\Delta(V'_1/V)$  and hence by Lemma 31, for any point  $P^*_1$  on  $V^*_1$  lying above  $P_1$ ,  $G_i(P^*_1/P_1)$  must be nonabelian. Now take  $S = Q(P_1, V)$ .

LEMMA 35. *Same as Lemma 34 without the assumption that  $R^*$  is the only local ring in  $K^*$  lying above  $R$ .*

*Proof.* Let  $K_1$  be the splitting field of  $R^*/R$ , let  $R_1 = R^* \cap K_1$ . Then by Lemma 34, there exists an immediate quadratic transform  $S_1$  of  $R_1$  such that for a local ring  $S^*$  in  $K^*$  lying above  $S_1$ ,  $G_i(S^*/S_1)$  is nonabelian. Let  $S = S_1 \cap K$ . Then by Proposition 1,  $S$  is an immediate quadratic transform of  $R$  and  $S_1$  lies above  $S$ . Hence by Lemma 2 of Part I,  $G_i(S^*/S)$  is nonabelian.

*Proof of Proposition 11 $\beta$ .* Assume, if possible, that  $G_i(R^*/R)$  is nonabelian. Then by Lemma 35 there exists a sequence  $R = R_0, R_1, R_2, \dots$  of successive immediate quadratic transforms such that for each  $n$  and for any local ring  $R^*_n$  in  $K^*$  lying above  $R_n$ ,  $G_i(R^*_n/R_n)$  is nonabelian. By [Abhyankar 5, Lemma 4.5 of Section 15],  $\bigcup_{i=0}^{\infty} R_i$  is the valuation ring of a zero dimensional valuation of  $K/k$  and hence as in the proof of [Abhyankar 5, Theorem 4.9 of Section 17] for some  $n$ ,  $G_i(R^*_n/R_n)$  is abelian.<sup>12</sup> This being a contradiction, Proposition 11 $\beta$  is proved.

*Proof of Proposition 12 $\beta$ .* This now follows from Proposition 11 $\beta$  and Lemma 30.

*Proof of Proposition 13 $\beta$ .* This now follows from Proposition 11 $\beta$  and Lemma 31.

Remark 5. Proposition 12 $\beta$  (and hence Proposition 12) is false if  $W_1$  is only required to have a normal crossing at  $P$  instead of a strong normal crossing. We shall illustrate this by the following example. For  $V$  take the projective plane over  $k$ , let  $X, Y$  be affine coordinates in  $V$ , let  $(x, y)$  be the corresponding general point of  $V/k$ , for  $V^*$  take the surface in projective

<sup>12</sup> The proof of the quoted theorem applies since the order of  $G_i(R^*/R)$  is prime to  $p$  in case  $p \neq 0$ .

three space with affine equation  $Z^2 = 1 + Y$ , let  $z$  be a root of  $Z^2 = 1 + y$ , and let  $\phi$  be the projection along the  $Z$ -axis. Assume that  $p \neq 2$ . Then  $V^*$  has no singularities at finite distance and hence it is normal at finite distance. Now  $K = k(x, y)$ ,  $K^* = k(x, y, z)$ ,  $\Delta(K^*/V) = L \cup L_\infty$  where  $L$  is the line:  $1 + Y = 0$  and  $L_\infty$  is the line at infinity. Let  $P$  be the point  $X = Y = 0$  and  $L_\infty$  is the line at infinity. Let  $P$  be the point  $X = Y = 0$ . Then above  $P$  lie the points  $P_1: X = Y = 0, Z = 1$  and  $P_2: X = Y = 0, Z = -1$ . Let  $f = y^3 + y^2 - x^2$ , and let  $W_1$  be the curve:  $Y^3 + Y^2 - X^2 = 0$ . Let  $W = L \cup L_\infty \cup W_1$ . Then  $\Delta(K^*/V) \subset W$  and  $W$  has a normal crossing at  $P$  (Lemma 17) and  $W_1$  is irreducible since  $Y^3 + Y^2$  is not a square. Let  $(R_1, M_1)$  be the quotient ring of  $P_1$  on  $V^*$ . Then  $R_1$  is a unique factorization domain and  $(x, y)$  is a minimal basis of  $M_1$ . Now

$$f = y^3 + y^2 - x^2 = y^2(1 + y) - x^2 = y^2z^2 - x^2 = (yz - x)(yz + x).$$

Since  $z$  is a unit in  $R_1$ , the reduced  $R_1$ -leading forms of  $(yz - x)$  and  $(yz + x)$  are coprime linear forms; hence  $H_1 = (yz - x)R_1$  and  $H_2 = (yz + x)R_1$  are distinct one dimensional prime ideals in  $R_1$  and  $fR_1 = H_1H_2$  and consequently  $\phi^{-1}(W_1)$  splits locally at  $P_1$  (and similarly  $\phi^{-1}(W_1)$  splits locally at  $P_2$ ).

*Remark 6 (Corrections to Part I).* We take this opportunity to correct some errors in Part I and to make some remarks concerning them. (1) In the third line before Proposition 1 on page 57 of Part I, the phrase " $\Delta(K^*/V)$  has a normal crossing at  $P$ " should be replaced by the phrase " $\Delta(K^*/V)$  is contained in a pure  $n-1$  dimensional subvariety  $W$  of  $V$  having a strong normal crossing at  $P$ ," and (2) in Proposition 2 on page 57 of Part I, the phrase "component of  $\Delta(K^*/V)$ " should be replaced by the phrase "component of  $W$ "; because as was shown in Remark 5 above, Proposition 2 of Part I would otherwise be false. In this modified form, Propositions 1 and 2 of Part I now follow respectively from Propositions 11 $\alpha$  and 12 $\alpha$  of the present paper. This modification of Proposition 2 of Part I now necessitates that the following additional changes be made in Part I: (3) Twice in the Introduction and once each in the statements of Propositions 6, 7, 8, 9 and Theorems 1, 2, 3, 4, 5 of Part I, the phrase "normal crossings" should be changed to the phrase "strong normal crossings." (4) Lines 8 and 9 in the Proof of Proposition 6 on page 74 of Part I should read " $P = \phi(P^*)$ . Now  $\Delta(K^*/V) \subset W$ ,  $W_i$  is an irreducible component of  $W$ , and  $W$  has a strong normal crossing at  $P$ ." (5) In the last third and fourth lines on page 75 of Part I, "Now . . .  $\Delta(K^*/V)$ ," should be replaced by "By (2),  $W$  has a strong normal crossing at  $P$ ,". (6) After line 8 on page 90 of Part I add "and assuming that the normal crossings are strong normal crossings,".

(7) Line 30 on page 77 of Part I should read "and hence  $u_{q+1}^{-1}(b_j^q)$  contains a generator of  $G^{q+1}$  and we take one such generator for  $b_j^{q+1}$ ." Note that this assertion now follows from Lemma 8 of the present paper. (8) Also, referring to the fourth sentence in the proof of Lemma 32 on page 80 of Part I, the existence of  $v$  stated therein follows from Lemma 8 of the present paper in view of the fact that all the  $m_q$ -th roots of unity form a multiplicative finite cyclic group. (9) In the third line in footnote 7 on page 61 of Part I, the word "normal" is to be omitted. (10) In the first line after Lemma 4 on page 53 of Part I, the second letter  $K$  should be replaced by the letter  $K^*$ . (11) In Theorem 3 on page 79 of Part I,  $V$  should be replaced by  $P_n$ . (12) In line 7 on page 88 of Part I, the second (i) should be replaced by (ii).

(13) Proofs of Lemmas 8 and 9 of Section 2 of Part I need some clarification in the nongalois cases. First note that Lemmas 8 and 9 were not used in the proofs of Lemmas 10 and 11 of that section, hence the latter may be used in the proving the former. Here we shall first prove Lemma 9 and then Lemma 8.<sup>13</sup>

*Proof of Lemma 9.* First assume that  $K_1/V$  is unramified. To show that  $K^*_1/V^*$  is unramified, in view of Lemma 4 it is enough to show that any point  $P^*$  of  $V^*$  is unramified in  $K^*_1$ . Let  $R^* = Q(P^*, V^*)$ , let  $R^*_1$  be a local ring in  $K^*_1$  lying above  $R^*$ , let  $R = R^*_1 \cap K$  and  $R_1 = R^*_1 \cap K_1$ . Let  $E, E_1, E^*, E^*_1$  be the quotient fields of the completions of  $R, R_1, R^*, R^*_1$  respectively. Then by Proposition 1 of [Abhyankar 2] we can conclude that  $E^*_1$  is a compositum of  $E_1$  and  $E^*$ , and hence  $[E^*_1 : E^*] \leq [E_1 : E]$ . Since the residue fields of  $R$  and  $R^*$  are algebraically closed (they are isomorphic to  $k$ ),  $r(R^*_1 : R^*) = [E^*_1 : E^*]$  and  $r(R_1 : R) = [E_1 : E]$ . Since  $K_1/V$  is unramified,  $r(R_1 : R) = 1$ . Therefore  $r(R^*_1 : R^*) = 1$  i.e.,  $R^*_1/R^*$  is unramified. This shows that  $K^*_1/V^*$  is unramified. Now assume that  $K^*_1/V$  is tamely ramified and keep the above notation. To show that  $K^*_1/V^*$  is tamely ramified, in view of Lemmas 2 and 7 we may assume that  $K_1/K$  is galois, and in view of Lemma 11 it is enough to show that  $R^*_1/R^*$  is tamely ramified. We know that  $[E_1 : E] = r(R_1/R) \not\equiv 0 \pmod{p}$  (for  $p=0$  there is nothing to show, so we are assuming that  $p \neq 0$ ). Let  $F$  be a galois extension of  $E$  containing  $E^*_1$ .  $K_1/K$  is galois implies  $E_1/E$  is galois [Abhyankar 2, Section 2], i.e.,  $G(F/E_1)$  is a normal subgroup of  $G(F/E)$ , and hence  $G(F/E_1) \cap G(F/E^*)$  is a normal subgroup of  $G(F/E^*)$ . Now  $[G(F/E) : G(F/E_1)] = [E_1 : E] \not\equiv 0 \pmod{p}$  and by the homomorphism theorem,

<sup>13</sup> In these proofs, references to various lemmas are to those in Part I.

$$G(F/E^*)/(G(F/E_1) \cap G(F/E^*))$$

is isomorphic to a subgroup of  $G(F/E)/G(F/E_1)$  and hence  $[G(F/E^*): G(F/E_1) \cap G(F/E^*)] \not\equiv 0 \pmod{p}$ . Now  $E^*_1$  is the compositum of  $E^*$  and  $E_1$  and hence  $G(F/E^*_1) = G(F/E_1) \cap G(F/E^*)$ . Therefore  $E^*_1/E^*$  is galois and  $r[E^*_1: E^*] = [E^*_1: E^*] \not\equiv 0 \pmod{p}$ . This completes the proof of Lemma 9. *Proof of Lemma 8.* In view of Lemmas 7 and 9, we may replace  $K_1$  by a least galois extension  $L$  of  $K$  containing  $K_1$  and replace  $K_2$  by the compositum of  $L$  and  $K_2$ , and then again replace  $K_2$  by a least galois extension of  $K_2$  containing  $K_1$ . Thus to begin with we may assume that  $K_1/K$  and  $K_2/K_1$  are galois. Let  $P$  be a point on  $V$ , let  $R = Q(P, V)$ , let  $R_1$  be a local ring in  $K_1$  lying above  $R$  and let  $R_2$  be a local ring in  $K_2$  lying above  $R_1$ . Let  $E, E_1, E_2$  be the quotient fields of the completions of  $R, R_1, R_2$  respectively and let  $E_3$  be a least galois extension of  $E$  containing  $E_2$ . Then  $[E_2: E_1] = r(R_2: R_1) \not\equiv 0 \pmod{p}$ ,  $[E_1: E] = r(R_1: R) \not\equiv 0 \pmod{p}$ , (for  $p=0$  there is nothing to prove, so we are assuming that  $p \neq 0$ ),  $E_2/E_1$  and  $E_1/E$  are galois [Abhyankar 2, Section 2]. In view of Lemma 11 it is enough to show that  $R_2/R$  is tamely ramified, i.e.,  $[E_3: E] \not\equiv 0 \pmod{p}$ . Let  $p^a$  be the highest power of  $p$  that divides  $[E_3: E_2]$ . Then  $p^a$  is also the highest power of  $p$  that divides  $[E_3: E]$  as well as the highest power of  $p$  that divides  $[E_3: E_1]$ . Therefore by the Sylow theorem,  $G(E_3/E_2)$  contains a subgroup  $H$  of order  $p^a$  and  $H$  is necessarily a  $p$ -Sylow subgroup of  $G(E_3/E)$ , of  $G(E_3/E_1)$ , and of  $G(E_3/E_2)$ . Now  $p$ -Sylow subgroups of  $G(E_3/E)$  are conjugates in  $G(E_3/E)$  and  $G(E_3/E_1)$  is a normal subgroup of  $G(E_3/E)$  and hence they are all contained in  $G(E_3/E_1)$  and hence they coincide with the  $p$ -Sylow subgroups of  $G(E_3/E_1)$ . For the same reason the  $p$ -Sylow subgroups of  $G(E_3/E_1)$  are also the  $p$ -Sylow subgroups of  $G(E_3/E_2)$ . Consequently the subgroup  $T$  of  $G(E_3/E)$  generated by all the  $p$ -Sylow subgroups of  $G(E_3/E)$  belongs to  $G(E_3/E_2)$ . Now  $T$  is a normal subgroup of  $G(E_3/E)$ ; however since  $E_3$  is a least galois extension of  $E$  containing  $E_2$ , the only normal subgroup of  $G(E_3/E)$  which is contained in  $G(E_3/E_2)$  is the identity subgroup. Therefore  $T=1$  and hence  $q=0$ . This completes the proof of Lemma 8.

**9. Main results on fundamental groups.** Throughout this section  $K$  will denote a two dimensional algebraic function field over an algebraically closed ground field  $k$  of characteristic  $p$ , and  $V$  will denote a nonsingular projective model of  $K/k$ ,  $W$  will denote a curve on  $V$ , and  $W_1, \dots, W_i$  will denote the irreducible components of  $W$ .

PROPOSITION 14. *Let  $K^*$  be a finite separable algebraic extension of  $K$  such that  $K^*/V$  is tamely ramified and  $\Delta(K^*/V) \subset W$ , let  $V^*$  be a  $K^*$ -normalization of  $V$  and let  $\phi$  be the rational map of  $V^*$  onto  $V$ . If  $\dim |W_1| > 1 + \nu(W_1, W; V)$  then  $\phi^{-1}(W_1)$  is irreducible.*

*Proof.* Let  $K'$  be a least galois extension of  $K$  containing  $K^*$ , let  $V'$  be a  $K'$ -normalization of  $V$ , and let  $g$  be the rational map of  $V'$  onto  $V$ . Then by Lemmas 5 and 7 of Section 2 of Part I,  $K'/V$  is tamely ramified and  $\Delta(K'/V) \subset W$ . By Proposition 5 of Section 6, there exists a quadratic transform  $(V_1, f)$  of  $V$  such that  $f^{-1}(W)$  has a strong normal crossing at each point of  $f^{-1}[W_1]$  and  $\dim |f^{-1}[W_1]| > 1$ . Let  $V'_1$  be a  $K'$ -normalization of  $V_1$ , and let  $g_1$  be the rational map of  $V'_1$  onto  $V_1$ . Then it is clear that  $\Delta(K'/V_1) \subset f^{-1}(W)$ , and by Lemma 6 of Section 2,  $K'/V_1$  is tamely ramified. Let  $v$  be the valuation of  $K/k$  having center  $W_1$  on  $V$ . Then  $v$  has center  $f^{-1}[W_1]$  on  $V_1$ . By Proposition 5 of Section 11 of Part I,  $g_1^{-1}(f^{-1}[W_1])$  is connected. Suppose if possible that  $g_1^{-1}(f^{-1}(W_1))$  is reducible, then we can find two distinct irreducible components  $H$  and  $L$  of  $g_1^{-1}(f^{-1}[W_1])$  which have a point  $P'$  in common. Let  $P = g_1(P')$ . Then  $P \in f^{-1}[W_1]$  and hence  $f^{-1}(W)$  has a strong normal crossing at  $P$ . Therefore by Proposition 12 $\beta$  of Section 8, only one irreducible component of  $g_1^{-1}(f^{-1}[W_1])$  can pass through  $P'$ , which is a contradiction. Therefore  $g_1^{-1}(f^{-1}[W_1])$  is irreducible and hence  $v$  has only one extension to  $K'$ . Therefore  $v$  has only one extension to  $K^*$  and hence  $\phi^{-1}(W_1)$  is irreducible.

PROPOSITION 15. *Assume that  $V$  is simply connected and  $\dim |W_j| > 1 + \nu(W_j, W; V)$  for  $j = 1, \dots, t$ . Let  $K^*/K$  be a galois extension such that  $K^*/V$  is tamely ramified and  $\Delta(K^*/V) \subset W$ . Let  $V^*$  be a  $K^*$ -normalization of  $V$  and let  $\phi$  be the rational map of  $V^*$  onto  $V$ . Then we have the following:*

(A)  $W^*_j = \phi^{-1}(W_j)$  is irreducible for  $j = 1, 2, \dots, t$ .

(B) The inertia group  $G_i(W^*_j/W_j)$  is a cyclic normal subgroup (of order prime to  $p$  in case  $p \neq 0$ ) of  $G(K^*/K)$  for  $j = 1, 2, \dots, t$ . Let  $a_j$  be generator of  $G_i(W^*_j/W_j)$ .

(C)  $G(K^*/K)$  is generated by

$$G_i(W^*_1/W_1), G_i(W^*_2/W_2), \dots, G_i(W^*_t/W_t). \text{ Hence}$$

(D)  $G(K^*/K)$  is generated by the  $t$  generators  $a_1, a_2, \dots, a_t$  each of which generates a normal subgroup.

(E)  $G(K^*/K)$  is  $t$ -step nilpotent and its order is not divisible by  $p$  in case  $p \neq 0$ .

(F) If  $W_j$  and  $W_k$  have a point in common at which  $W$  has a normal crossing then  $a_j$  and  $a_k$  commute in  $G(K^*/K)$ .

(G) If  $W_j$  and  $W_k$  have a point in common at which  $W$  has a normal crossing whenever  $j \neq k$ , then  $G(K^*/K)$  is abelian.

*Proof.* Follows from the proof of Theorem 1 of Section 11 of Part I after replacing the reference there to Propositions 6 and 1 of Part I by reference to Propositions 14 and 11 $\beta$  of the present paper.<sup>14</sup>

**THEOREM 1.** Assume that  $V$  is simply connected and  $\dim |W_j| > 1 + v(W_j, W; V)$  for  $j = 1, \dots, t$ . Then we have the following:

(A)  $V - W$  has a tame fundamental weak parent group  $G$  generated by  $t$  generators  $a_1, a_2, \dots, a_t$  with a weak parent map  $f$  of  $G$  onto  $\pi'(V - W)$  such that in each member  $H$  of  $\pi'(V - W)$ ,  $a_j$  (i.e., the  $f$  image of  $a_j$ ) generates the cyclic (and normal in  $H$ ) inertia group over  $W_j$  of the unique irreducible curve corresponding to  $W_j$  on a normalization of  $V$  in the galois extension of  $K$  corresponding to  $H$ ; also  $a_j$  and  $a_k$  commute in  $G$  if  $W_j$  and  $W_k$  have a point in common at which  $W$  has a normal crossing.

(B) Every unrestricted tame fundamental weak parent group<sup>15</sup> of  $V - W$  is  $t$ -step nilpotent.

(C) If  $W_j$  and  $W_k$  have a point in common at which  $W$  has a normal crossing whenever  $j \neq k$  then every unrestricted tame fundamental weak parent group<sup>15</sup> of  $V - W$  is abelian.

(D)  $\pi^*(V - W) = \pi'(V - W)$ .

*Proof.* Follows from the proof of<sup>16</sup> Theorem 2 of Section 12 of Part I after replacing the reference there to Theorem 1 of Part I by the reference to Proposition 15 above.

**PROPOSITION 16.** Assume that  $V$  is simply connected and the irreducible components  $W_j$  can be labelled so that

<sup>14</sup> The adjective "normal crossing" in the quoted results of Part I is to be corrected to "strong normal crossing"; see Remark 6 of Section 8 of the present paper.

<sup>15</sup> In particular,  $G$  and the inverse limit of  $\pi'(V - W)$ , i.e., the galois group over  $K$  of the compositum of all the members of  $\Omega'_\rho(V - W)$ , as well as every unrestricted tame fundamental parent group of  $V - W$ .

<sup>16</sup> Note correction (7) given in Remark 6 of Section 8.

$$\dim |W_j| > v(W_j, W_j \cup W_{j+1} \cup \cdots \cup W_t)$$

for  $j=1, \dots, t$ . Let  $K^*/K$  be a galois extension such that  $K^*/V$  is tamely ramified and  $\Delta(K^*/V) \subset W$ . Let  $V^*$  be a  $K^*$ -normalization of  $V$  and let  $\phi$  be the rational map of  $V^*$  onto  $V$ . Choose an irreducible component  $W^*_j$  of  $\phi^{-1}(W_j)$ . Let  $H_j$  be the subgroup of  $G(K^*/K)$  generated by  $G_i(W^*_1/W_1), \dots, G_i(W^*_j/W_j)$ , let  $K_j$  be the fixed field; let  $V_j$  be a  $K_j$ -normalization of  $V$  and let  $\phi_j$  be the rational map of  $V_j$  onto  $V$ ; also set  $H_0=1$ ,  $K_0=K^*$ ,  $V_0=V^*$  and  $\phi_0=\phi$ . Then we have the following:

(A)  $\phi_j^{-1}(W_{j+1})$  is irreducible for  $j=0, 1, \dots, t-1$ .

(B)  $H_j$  is a normal subgroup of  $G(K^*/K)$ , i.e.,  $K_j/K$  is galois for  $j=1, \dots, t$ ; and  $H_t=G(K^*/K)$ , i.e.,  $K_t=K$ . Let  $\alpha_j$  be the canonical homomorphism of  $G(K^*/K)$  onto  $G(K_j/K)$ .

(C)  $\alpha_j(H_{j+1}) = G_i(\phi_j^{-1}(W_{j+1})/W_{j+1})$  for  $j=0, 1, \dots, t-1$ .

(D)  $G_i(W^*_j/W_j)$  and  $G_i(\phi_j^{-1}(W_j)/W_j)$  are cyclic for  $j=1, 2, \dots, t$ ; and if  $a_j$  is an element of  $G(K^*/K)$  such that  $\alpha_j(a_j)$  generates  $G_i(\phi_j^{-1}(W_j)/W_j)$ , in particular if  $a_j$  is a generator of  $G_i(W^*_j/W_j)$ , then  $a_1, \dots, a_j$  generate the normal subgroup  $H_j$  of  $G(K^*/K)$  for  $j=1, \dots, t$ ; and  $a_1, \dots, a_t$  generate  $G(K^*/K)$ .

(E)  $G(K^*/K)$  is  $t$ -step solvable and its order is prime to  $p$  in case  $p \neq 0$ .

*Proof.* (C), (D) and (F) follow at once from (A) and (B) in view of Lemmas 1 and 14 of Section 2 of Part I. We shall prove (A) and (B) by induction on  $t$ . For  $t=1$  this reduces to Proposition 15, so now assume that  $t > 1$  and that (A) and (B) are true for  $t-1$ . By Proposition 14,  $\phi_0^{-1}(W_1)$  is irreducible so that  $\phi_0^{-1}(W_1)=W^*_1$  and by Lemma 1 of Section 2 of Part I,  $H_1=G_i(W^*_1/W_1)$  is a normal subgroup of  $G(K^*/K)$ . Let  $\phi'$  be the rational map of  $V^*$  onto  $V_1$ , and let  $W'_j=\phi'(W^*_j)$ . Then  $W'_j$  is an irreducible component of  $\phi_1^{-1}(W_j)$  and by Lemma 2 of Part I,  $\alpha_1(H_j)$  is generated by  $G_i(W'_2/W_2), \dots, G_i(W'_j/W_j)$ . By Lemmas 1 and 6 of Part I,  $W_1$  is not ramified in  $K_1$  and hence by Lemma 12 of Part I and Proposition 13 $\beta$ ,  $\Delta(K_1/V) \subset W_2 \cup \dots \cup W_t$  and  $K_1/V$  is tamely ramified. Hence the induction hypothesis is satisfied if we replace  $K^*$  by  $K_1$  and  $W$  by  $W_2 \cup \dots \cup W_t$ ; hence we have that  $\phi_j^{-1}(W_{j+1})$  is irreducible and  $\alpha_1(H_j)$  is a normal subgroup of  $G(K_1/K)$  for  $j=1, \dots, t-1$ , and that  $\alpha_1(H_t)=G(K_1/K)$ . From this (A) and (B) follow at once. Thus the induction is complete and the proposition is proved.

THEOREM 2. Assume that  $V$  is simply connected and that the irreducible components  $W_j$  can be labelled so that  $\dim |W_j| > 1 + v(W_j, W_j \cup \dots \cup W_t)$  for  $j=1, \dots, t$ . Then we have the following:

(A)  $V-W$  has a tame fundamental weak parent group  $G$  generated by  $t$  generators  $a_1, \dots, a_t$  with a weak parent map  $f$  of  $G$  onto  $\pi'(V-W)$  such that for each member  $H$  of  $\pi'(V-W)$  after denoting the corresponding member of  $\Omega'_g(V-W)$  by  $K^*$ , a  $K^*$ -normalization of  $V$  by  $V^*$  and the rational map of  $V^*$  onto  $V$  by  $\phi$  we have the following: (1) for some irreducible component  $W^*_j$  of  $\phi^{-1}(W_j)$ ,  $f_H(a_j)$  is a generator of  $G_i(W^*_j/W_j)$  for  $j=1, \dots, t$ ; (2)  $a_1, \dots, a_j$  generate a normal subgroup  $H_j$  of  $H = G(K^*/K)$  for  $j=1, \dots, t$  and  $H_t = H$ ; (3) if we denote the fixed field of  $H_j$  by  $K_j$ , a  $K_j$ -normalization of  $V$  by  $V_j$ , the rational map of  $V_j$  onto  $V$  by  $\phi_j$ , the natural homomorphism of  $H$  onto  $G(K_j/K)$  by  $\alpha_j$  and  $H$  by  $H_0$ , then  $\phi_j^{-1}(W_{j+1})$  is irreducible and  $\alpha_j(H_{j+1}) = G_i(\phi_j^{-1}(W_{j+1})/W_{j+1})$  for  $j=0, 1, \dots, t-1$ .

(B) Every unrestricted tame fundamental weak parent group<sup>15</sup> of  $V-W$  is  $t$ -step solvable.

(C)  $\pi^*(V-W) = \pi'(V-W)$ .

*Proof.* Replacing the reference to<sup>14</sup> Theorem 1 of Part I by reference to the above Proposition 16, from any one of the three proofs (i), (ii), (iii) given in Theorem 2 of Section 12 of Part I of the initial italicized assertion in the proof of that theorem we conclude that  $\pi'(V-W)$  contains an ascending cofinal sequence,  $1 = G^1 < G^2 < \dots$ . Let  $K = K^1 < K^2 < \dots$  be the corresponding galois extension of  $K$  with  $G(K^q/K) = G^q$ . Let  $v_j = v_j^1$  be the real discrete valuation of  $K/k$  having center  $W_j$  on  $V$ . Fix an arbitrary  $K^2$ -extension  $v_j^2$  of  $v_j^1$ , then fix an arbitrary  $K^3$ -extension  $v_j^3$  of  $v_j^2$ , and so on, thus getting a  $K^q$ -extension  $v_j^q$  of  $v_j$  for all  $q$  such that  $v_j^b$  is a  $K^b$ -extension of  $v_j^a$  whenever  $a < b$ . By Lemmas 2 and 14 of Section 2 of Part I, for all  $q$  we have that  $G_i(v_j^q/v_j)$  is cyclic and  $u_{q+1}(G_i(v_j^{q+1}/v_j)) = G_i(v_j^q/v_j)$  where  $u_{q+1}$  denotes the natural homomorphism of  $G(K^{q+1}/K)$  onto  $G(K^q/K)$ . In view of Lemma 8 of Section 2, by induction on  $q$  we can fix a generator  $b_j^q$  of  $G_i(v_j^q/v_j)$  such that  $u_{q+1}(b_j^{q+1}) = b_j^q$  for all  $q$ . Then by Proposition 16,  $b_1^q, \dots, b_t^q$  generate  $G^q = G(K^q/K)$  for each  $q$ . Since  $G^1 < G^2 < \dots$  is cofinal in  $\pi'(V-W)$ , by Lemma 22 of Section 8 of Part I, we can find a group  $G$  generated by  $t$  generators  $a_1, \dots, a_t$  and a weak parent map  $f$  of  $G$  onto  $\pi'(V-W)$  such that the  $f$  image of  $a_j$  in  $G^q$  is  $b_j^q$  for all  $q$  and for  $j=1, \dots, t$ . In view of Lemma 2 of Section 2 of Part I and Proposition 4 of Section 9 of Part I, everything now follows from the above Proposition 16.

*Remark 7.* Remarks 8, 9, and 11 of Section 12 of Part I now apply after replacing reference there to Theorem 2 of Part I by reference to the above Theorems 1 and 2 and replacing the reference there to Theorem 1 of Part I by reference to the above Propositions 15 and 16. Also note that for dimension two, Theorems 1 and 2 of Part I are now subsumed respectively under Proposition 15 and Theorem 1 above.<sup>14</sup>

*Remark 8.* It is clear that Theorems 1 and 2 together with their proofs can be combined in various ways to get other results, here we shall give three examples of modified assumptions on  $W$  leaving the conclusions on  $\pi'(V - W)$  to the reader: (1) Assume that the components  $W_j$  can be labelled so that

$$\dim |W_j| > 1 + \nu(W_j, W; V) \text{ for } j = 1, 2, \dots, s,$$

$$\dim |W_j| > 1 + \nu(W_j, W_j \cup W_{j+1} \cup \dots \cup W_t)$$

$$\text{for } j = s+1, s+2, \dots, t,$$

and that  $W_j$  and  $W_k$  have a point in common at which  $W$  has a strong normal crossing whenever  $j, k \leq s$  and  $j \neq k$ . (2) Assume that the components  $W_j$  can be labelled so that

$$\dim |W_j| > 1 + \nu(W_j, W_j \cup W_{j+1} \cup \dots \cup W_t; V) \text{ for } j = 1, \dots, s$$

and

$$\dim |W_j| > 1 + \nu(W_j, W_{s+1} \cup W_{s+2} \cup \dots \cup W_t; V)$$

$$\text{for } j = s+1, s+2, \dots, t.$$

(3) Assume that the components  $W_j$  can be labelled so that

$$\dim |W_j| > 1 + \nu(W_j, W_j \cup W_{j+1} \cup \dots \cup W_t; V) \text{ for } j = 1, \dots, s;$$

$$\dim |W_j| > 1 + \nu(W_j, W_{s+1} \cup W_{s+2} \cup \dots \cup W_t; V)$$

$$\text{for } j = s+1, s+2, \dots, t;$$

and  $W_j$  and  $W_k$  have a point in common at which  $W_{s+1} \cup W_{s+2} \cup \dots \cup W_t$  has a strong normal crossing whenever  $j, k > s$  and  $j \neq k$ .

**10. Applications.** Throughout this section  $k$  will denote an algebraically closed ground field of characteristic  $p$ . Note that the dimension of the complete linear system determined by a curve of degree  $g^*$  in a projective plane over  $k$  is given by  $(g^{*+2}) - 1 = \frac{1}{2}g^*(g^* + 3)$ .

**THEOREM 3.** Let  $W$  be a curve in the projective plane  $P_2$  over  $k$ ; let  $W_1, \dots, W_t$  be the irreducible components of  $W$ ; let  $g^*_j$  be the degree of  $W_j$

and let  $d=1$  in case  $p=0$  and  $d$  the highest power of  $p$  which divides  $g^*_1, \dots, g^*_t$  in case  $p \neq 0$ ; let  $g_j = g^*_j d^{-1}$ , and let  $G$  be the abelian group on  $t$  generators  $a_1, \dots, a_t$  with the only relation

$$a_1^{g_1} \cdots a_t^{g_t} = 1.$$

Assume that  $\frac{1}{2}g^*_j(g^*_j + 3) > 1 + v(W_j, W; P_2)$  for  $j=1, \dots, t$ , and that  $W_j$  and  $W_k$  have a point in common at which  $W$  has a strong normal crossing whenever  $j \neq k$ . Then  $G$  is a tame fundamental parent group of  $P_2 - W$ . Also  $\pi^*(P_2 - W) = \pi'(P_2 - W)$  and hence  $G$  is a reduced fundamental parent group of  $P_2 - W$  as well.  $G$  is a direct product of a free abelian group on  $t-1$  generators and a cyclic group of order equal to the greatest common divisor of  $g_1, g_2, \dots, g_t$ ; i.e., equal to the greatest common divisor of  $g^*_1, g^*_2, \dots, g^*_t$  in case  $p=0$  and to the part of this prime to  $p$  in case  $p \neq 0$ .

*Proof.* Follows from the proof of Theorem 3 of Section 13 Part I after replacing the reference there to Theorem 1 of Part I by reference to Proposition 15 of Section 9 of the present paper.<sup>14</sup>

**THEOREM 4.** Let  $K$  be a two dimensional algebraic function field over  $k$ , let  $V$  be a nonsingular projective model of  $K/k$  and let  $W$  be an irreducible curve on  $V$ . Assume that  $V$  is simply connected and  $\dim |W| > 1 + v(W, W; V)$ . Let  $K^*$  be the compositum of all the fields in  $\Omega'_g(V - W)$ . Then (i)  $K^*/K$  is cyclic of degree  $\delta(W, V)$ ; and  $V - W$  has as a tame (as well as reduced) fundamental parent group a cyclic group of order  $\delta(W, V)$ . Let  $V^*$  be a  $K^*$ -normalization of  $V$  and let  $\phi$  be the map of  $V^*$  onto  $V$  (so that  $V^* - \phi^{-1}(W)$  is the "tame universal covering" of  $V - W$ ). Then (ii)  $V^* - \phi^{-1}(W)$  is tamely simply connected. Finally (iii) the normalization of  $V$  in any field between  $K$  and  $K^*$  (in particular  $V^*$ ) is simply connected.

*Proof.* Follows from the proof of Theorem 4 of Section 14 of Part I after replacing the reference there to Theorem 1 and Proposition 6 of Part I, respectively, by reference to Propositions 15 and 14 of Section 9 of the present paper.<sup>14</sup>

**PROPOSITION 17.** Let  $V^*$  be a normal projective surface over  $k$ . Assume that there exists a rational map  $\phi$  of  $V^*$  onto a nonsingular projective simply connected surface  $V$  of finite index such that: (1)  $\phi$  and  $\phi^{-1}$  are both free from fundamental points, (2)  $V^*/V$  is tamely ramified, (3)  $\Delta(V^*/V)$  is irreducible, and (4)  $\dim |\Delta(V^*/V)| > 1 + v(\Delta(V^*/V), \Delta(V^*/V); V)$ . Then  $V^*$  is simply connected,  $k(V^*)/k(V)$  is galois with galois group cyclic of order dividing  $\delta(\Delta(V^*/V), V)$ , and  $V^* - \phi^{-1}(\Delta(V^*/V))$  is tamely simply connected in case  $[k(V^*) : k(V)] = \delta(\Delta(V^*/V), V)$ .

*Proof.* This is essentially Theorem 4 stated from a covering to the projection instead of the other way around.

**PROPOSITION 18.** *Let  $W$  be an irreducible curve of reduced degree  $g$  and degree  $g^*$  in the projective plane  $P_2$  over  $k$  such that  $\frac{1}{2}g^*(g^* + 3) > 1 + \nu(W, W; V)$ . Let  $K^*$  be the compositum of all the fields in  $\Omega'_g(P_2 - W)$ . Then (i)  $K^*/k(P_2)$  is cyclic of degree  $g$  so that  $P_2 - W$  has for a tame (as well as reduced) fundamental parent group a cyclic group of order  $g$ . Let  $V^*$  be a  $K^*$ -normalization of  $P_2$  and let  $\phi$  be the rational map of  $V^*$  onto  $P_2$ . Then (ii)  $V^* - \phi^{-1}(W)$  is tamely simply connected. Finally (iii) the normalization of  $P_2$  in any field between  $k(P_2)$  and  $K^*$  (in particular  $V^*$ ) is simply connected.*

*Proof.* Follows from Theorem 4 in view of Lemma 34 of Section 13 of Part I or alternatively (i) is exactly Theorem 3 for  $t=1$  and (ii) and (iii) follow from (i) as in the proof of Theorem 4 of Part I.<sup>14</sup>

**THEOREM 5.** *Let  $V^*$  be a surface in projective 3 dimensional space  $P_3$  over  $k$  having an affine equation*

$$X_3^m - f(X_1, X_2) = 0$$

*where  $W: f(X_1, X_2) = 0$  is an irreducible curve of degree  $g^*$  and reduced degree  $g$  (i.e.,  $f$  is an irreducible polynomial of degree  $g^*$ ) in the projective plane  $P_2$  over  $k$  (with affine coordinates  $X_1, X_2$ ) such that  $\frac{1}{2}g^*(g^* + 3) > 1 + \nu(W, W; P_2)$  and such that  $m$  divides  $g$ . Then  $V^*$  is simply connected. If  $m=g$  then  $V^* - (f(X_1, X_2) = 0 \cap V^*)$  is tamely simply connected.*

*Proof.* Follows from the proof of Theorem 5 of Section 14 of Part I after replacing the reference there to Propositions 7 and 8 of Part I, respectively, by above Propositions 17 and 18 and substituting 2 for  $n$ .<sup>14</sup>

**PROPOSITION 19.** *Let  $V$  be a nonsingular algebraic surface over the complex ground field such that  $V$  has no finite unramified topological coverings (this is so in particular if  $\pi_1(V) = 1$ ), let  $W$  be a curve on  $V$  and let  $W_1, \dots, W_t$  be the irreducible components of  $W$ , and by  $\gamma_{\pi_1}(V - W)$  denote the factor group of  $\pi_1(V - W)$  by the intersection of all subgroups of finite index in  $\pi_1(V - W)$ . (1) If for some labelling of the components  $W_j$ ,  $\dim |W_j| > 1 + \nu(W_j, W_j \cup W_{j+1} \cup \dots \cup W_t; V)$  for  $j=1, \dots, t$ , then  $\gamma_{\pi_1}(V - W)$  is  $t$ -step solvable and it is an unrestricted tame fundamental parent group of  $V - W$  and its Krull-completion has a dense subgroup generated by  $t$  generators. (2) If  $\dim |W_j| > 1 + \nu(W_j, W; V)$  for  $j=1, \dots, t$  then  $\gamma_{\pi_1}(V - W)$  is  $t$ -step nilpotent. (3) If  $\dim |W_j| > 1 + \nu(W_j, W_j, V)$*

for  $j=1, \dots, t$  and  $W_j$  and  $W_k$  have a point in common at which  $W$  has a normal crossing whenever  $j \neq k$ , then  $\gamma\pi_1(V-W)$  is abelian.

*Proof.* Follows from Theorems 1 and 2 of Section 9 in view of the considerations of Section 15 of Part I.

*Remark 9.* Theorems 3, 4, 5 and Propositions 7, 8, 9 of Part I for  $n=2$  are now subsumed, respectively, under above Theorems 3, 4, 5 and Propositions 17, 18, 19. In deducing Theorem 3 we have only used a part of Theorem 1. Using the full force of Theorems 1 and 2 we could have stated the corresponding versions for the projective plane, we have not done this here because elsewhere we shall state stronger conclusions under these circumstances.

*Remark 10.* It is clear that using Propositions 6 to 10 of Section 7 we can get several explicit corollaries for all the results of this and the previous section. Let us illustrate this by some examples.

*Example 1 (For Theorem 1).* Let  $V$  be a nonsingular simply connected algebraic surface over  $k$ , let  $W$  be a curve on  $V$ , and let  $W_1, \dots, W_t$  be the irreducible components of  $W$ . Assume that at a common point of distinct irreducible components of  $W$  only two irreducible components pass and each has a simple point there, and let  $j_a$  be the sum of orders of contact of  $W_j$  with the various  $W_i$  ( $i \neq j$ ) at the various points of intersection. Also assume that for each  $j$ , the singularities of  $W_j$  are all cusps and let  $j_1, \dots, j_b$  be the orders of these cusps. If

$$\dim |W_j| > 1 + j_a + \frac{1}{2} \sum_{q=1}^{b_j} j_q(j_q + 3)$$

for  $j=1, \dots, t$ , then  $\pi'(V-W) = \pi^*(V-W)$  is generated by  $t$  generators and is  $t$ -step nilpotent; if in addition  $W_j$  and  $W_i$  have a point in common at which  $W$  has a normal crossing whenever  $j \neq i$ , then  $\pi'(V-W)$  is abelian.

*Example 2 (For Proposition 18).* Let  $W$  be an irreducible curve in the projective plane  $P_2$  over  $k$ . Let  $g$  be the reduced degree of  $W$  and let  $g^*$  be the degree of  $W$ . Assume that any point  $P$  of  $P_2$  at most two analytic branches of  $W$  have a common tangent and that if two analytic branches of  $W$  at  $P$  do have a common tangent then each of these branches have a simple point at  $P$ ; also assume that at any point  $P$  of  $W$  all analytic branches of  $W$  not having a simple point at  $P$  necessarily have a cusp at  $P$ . Let  $u_1, \dots, u_a$  denote the orders of contact of various pairs of branches having tangents at various points of  $W$  ( $u_i > 1$ ); let  $v_1, \dots, v_b$  denote the orders of the cusps of the

various branches of  $W$  at various points of  $W$  ( $v_i > 1$ ), and let  $w_1, \dots, w_c$  denote the multiplicities of  $W$  at the various singular points of  $W$  ( $w_i > 1$ ). Denote by  $\pi'(P_2 - W)$  the galois group over  $k(P_2)$  of the compositum of all finite extensions of  $k(P_2)$  (in a fixed algebraic closure of  $k(P_2)$ ) which are tamely ramified over  $P_2$  and for which the branch locus on  $P_2$  is contained in  $W$ . If

$$\frac{1}{2}g^*(g^* + 3) > 1 + 3 \sum_{i=1}^a (u_i - 1) + \sum_{i=1}^b v_i + \frac{1}{2} \sum_{i=1}^c w_i(w_i + 1)$$

then  $\pi'(P_2 - V)$  is cyclic of order  $g$ .

*Example 3 (For Proposition 18). Irreducible plane curves of degree  $\leq 4$ .* Let  $W$  be an irreducible curve of degree  $g^*$  in the projective plane  $P_2$  over  $k$  and let  $\pi'(P_2 - W)$  denote the galois group over  $k(P_2)$  of the compositum of all finite galois extensions of  $k(P_2)$  (in some fixed algebraic closure of  $k(P_2)$ ) which are tamely ramified over  $P_2$  and for which the branch locus on  $P_2$  is contained in  $W$ . Here we consider the situation  $g^* \leq 4$ .

$g^* = 1$  (*Lines*).  $\dim |W| = 2$  and  $\nu(W, W; P_2) = 0$ . Hence  $\pi'(P_2 - W) = 1$ .

$g^* = 2$  (*Conics*).  $\dim |W| = 5$ . Again  $W$  is nonsingular and therefore  $\nu(W, W; P_2) = 0$ . Hence  $\pi'(P_2 - W)$  is cyclic of order 2 or 1 according as  $p \neq 2$  or  $p = 2$ .

$g^* = 3$  (*Cubics*).  $\dim |W| = 9$ .  $W$  can have at most one singularity, because otherwise for a line  $L$  joining two singularities of  $W$  we would have  $i(L \cdot W, P_2) > 3$ . Let  $P$  be a singularity of  $W$ . Then  $P$  must be a double point of  $W$ , because otherwise for a line  $L$  tangent to  $W$  at  $P$  we would have  $i(L \cdot W, P_2) > 3$ . If there are two branches of  $W$  at  $P$  then again for the same reason these two branches cannot have a common tangent and then  $\nu(W, W; P, V) = 3$ . If there is only one branch of  $W$  at  $P$  then for the tangent line  $L$  to  $W$  at  $P$  we must have  $\nu(L \cdot W; P, P_2) = 3$  and hence  $P$  must be a two-fold cusp of  $W$  and then  $\nu(W, W; P, P_2) = 5$ . Thus always  $\nu(W, W; P_2) + 1 \leq 5 + 1 = 6 < 9$ . Hence  $\pi'(P_2 - V)$  is cyclic of order 3 or 1 according as  $p \neq 3$  or  $p = 3$ .

$g^* = 4$  (*Quartics*). We shall show in a later paper that (1) if  $W$  does not have three cusps then  $\pi'(P_2 - W)$  is cyclic of order 4 or 1 according as  $p \neq 2$  or  $p = 2$ ; (2) any two quartics with three cusps are projectively equivalent; and (3) if  $W$  is an irreducible quartic having three cusps and  $p \neq 2, 3$  then  $\pi'(V - W)$  is a nonabelian group of order 12 and that this

group is the same as the one found by Zariski in the classical case [Zariski 11; Zariski 12, page 164].

*Remark 11.* In the situations of Theorems 1 and 2 of Section 9; conjecture 2 of Section 16 of Part I has now been verified and in the situation of Theorem 3 of this section, conjecture 1 of Section 16 of Part I has now been verified. Also Remark 14 of Section 17 of Part I now applies to the results of this and the previous section.

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# TAME COVERINGS AND FUNDAMENTAL GROUPS OF ALGEBRAIC VARIETIES.\*

## Part III: Some Other Sets of Conditions for the Fundamental Group to be Abelian.

By SHREERAM ABHYANKAR.

*Dedicated to my teacher Professor Oscar Zariski  
on his sixtieth birthday.*

**Introduction.** Let  $K$  be an  $n$  dimensional algebraic function field over an algebraically closed ground field  $k$  of characteristic  $p$ , and let  $V$  be a simply connected nonsingular projective model of  $K/k$ , let  $W$  be a pure  $(n-1)$ -dimensional subvariety of  $W$  and let  $W_1, \dots, W_t$  be the irreducible components of  $W$ . Denote by  $\pi'(V-W)$  the group tower of the galois groups over  $k(V)$  of all the finite galois extensions of  $k(V)$  (in some fixed algebraic closure of  $k(V)$ ) which are tamely ramified over  $V$  and for which the branch locus over  $V$  is contained in  $W$ .

In Theorem 2 of Section 12 of Part I<sup>1,2</sup> we proved that if (i)  $W$  has only strong normal crossings (ii)  $\dim |W_j| > 1$  for each  $j$  and (iii) the components  $W_j$  are pairwise connected, then  $\pi'(V-W)$  is abelian with  $t$  generators; also in Theorem 1 of Section 9 of Part II we proved that if (i)  $n=2$ , (ii)<sup>3</sup>  $\dim |W_j| > 1 + \nu(W_j, W, V)$  for each  $j$  and (iii)  $W_j$  and  $W_k$  have a point in common at which  $W$  has a normal crossing whenever  $j \neq k$ ; then  $\pi'(V-W)$  is abelian with  $t$  generators. In this paper we want to show that conditions (iii) above can be replaced by the condition that for each  $j$

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<sup>1</sup> "Part I: Branch loci with normal crossings; Applications: Theorems of Picard and Zariski," and "Part II: Branch curves with higher singularities," published in respectively in volumes 81 (1959) and 82 (1960) of this Journal; these will be referred to as "Part I" and "Part II" respectively. Notations, conventions and definitions given in Parts I and II will be followed and some more will be introduced in Section 1.

<sup>2</sup> Also see the correction given in Remark 6 of Section 8 of Part II.

<sup>3</sup>  $\nu(W_j, W; V)$  is a certain measure of the singularities of  $W_j$  relative to  $W$ ; for definition see Section 6 of Part II.

and  $k$  some nonzero integral multiple of  $W_j$  is linearly equivalent to some integral multiple of  $W_k$  (Theorem 1 of Section 3), and then in the second mentioned result condition (ii) can be replaced by the weaker condition that for some labelling of the components  $W_j$ ,

$$\dim |W_j| > 1 + v(W_j, W_j \cup W_{j+1} \cup \dots \cup W_t; V)$$

for each  $j$ ; this will be done by using the method of "removing tame ramification through cyclic compositum." (Proposition 1 of Section 2). Consequently in case  $V$  is the  $n$  dimensional projective space over  $k$  then we can drop conditions (iii) altogether, thus yielding a further generalization of Zariski's theorem [Part I, Theorem 3 of Section 13; Part II, Theorem 3 of Section 10] (Theorem 2 of Section 4).

**1. Notations and conventions.** Let  $v$  be a real discrete valuation of a field  $K$ , let  $p$  be the characteristic of the residue field of  $v$ , let  $K^*$  be a finite separable extension of  $K$ , and let  $v^*$  be a  $K^*$ -extension of  $v$ . In [A1, Section 1; A2, Section 2; Part I, Section 2]<sup>4</sup> we developed various notions of ramification and galois theories of quotient rings on normal algebraic varieties; the corresponding notions for real discrete valuations are well known and they also follow from the quoted reference since the only special properties of a normal local domain when it is the quotient ring on an algebraic variety which were used were that it is noetherian and its completion is also a normal local domain. Consequently we may and we shall use the notions, notations and results given in the above reference also for real discrete valuations. Then if  $\bar{R}_v, \bar{R}_v$  are the completion of  $R_v, R_v$  respectively and if  $E^*$  and  $E$  are the quotient fields  $\bar{R}_v, \bar{R}_v$  respectively then

$$\begin{aligned} d(v^*: v) &= \text{degree of } v^* \text{ over } v \\ &= d(R_v^*: R_v) = \text{degree of } R_v^* \text{ over } R_v \\ &= d(v^*: K) = \text{degree of } v^* \text{ over } K \\ &= [E^*: E]. \end{aligned}$$

$$\begin{aligned} g(v^*: v) &= \text{separable residue degree of } v^* \text{ over } v \\ &= g(R_v^*: R_v) = \text{separable residue degree of } R_v^* \text{ over } R_v \\ &= g(v^*: K) = \text{separable residue degree of } v^* \text{ over } K \\ &= [(R_v^*/M_v^*) : (R_v/M_v)]_s. \end{aligned}$$

$$\begin{aligned} i(v^*: v) &= \text{inseparable residue degree of } v^* \text{ over } v = \text{etc.} \\ &= [(R_v^*/M_v^*) : (R_v/M_v)]_i, \text{ in case } p \neq 0 \text{ and } 1 \text{ in case } p = 0. \end{aligned}$$

<sup>4</sup> Numbers in square brackets refer to the references at the end of the paper.

$$\begin{aligned} r(v^*: v) &= \text{ramification index of } v^* \text{ over } v = \text{etc.} \\ &= d(v^*: v)g(v^*: v)^{-1}. \end{aligned}$$

$$\begin{aligned} \bar{r}(v^*: v) &= \text{reduced ramification index of } v^* \text{ over } v = \text{etc.} \\ &= r(v^*: v)i(v^*: v)^{-1}. \end{aligned}$$

Now it follows from well known results [for instance C, Theorem 23] that the reduced ramification index of  $v^*$  over  $v$  is the index of the value group of  $v$  in the value group of  $v^*$ . Hence in particular if  $v^*_1, \dots, v^*_t$  are distinct extensions of  $v$  to  $K^*$  then  $\sum_{j=1}^t \bar{r}(v^*_j: v)g(v^*_j: v)i(v^*_j: v) = [K^*: K]$ . Note that if  $R_v$  is the quotient ring of an irreducible one codimensional subvariety  $W$  of an algebraic variety then the ramification and galois theoretic notions for  $v$  coincide with those for  $W$ .

For a polynomial  $f = f(X_1, \dots, X_n)$  we denote by  $\Delta_{X_1} f = \Delta_{X_1} f(X_1, \dots, X_n)$  the discriminant of  $f$  when  $f$  considered as a polynomial in  $X_1$  whose coefficients are polynomials in  $X_2, \dots, X_n$ ; when the reference is clear from the context, the subscript  $X_1$  may be dropped.

**2. Removing tame ramification through cyclic compositums.** In [A1, A3] and Parts I and II we have several times used a method which can be called "removing the tame ramification of a real discrete valuation through compositums with a cyclic extension," we give this in a general form in Proposition 1 below and it then subsumes Lemma 6 of [A1] and Proposition 8 of [A3].

**LEMMA 1.** *Let  $R$  be either the quotient ring of an irreducible subvariety of a normal algebraic variety or the valuation ring of a real discrete valuation. Let  $K$  be the quotient field of  $R$ , let  $L$  and  $K^*$  be finite separable extensions of  $K$ , let  $L^*$  be a compositum of  $L$  and  $K^*$ , let  $S^*$  be a local ring in  $L^*$  lying above  $R$ , let  $R^* = K^* \cap S^*$  and  $S = L \cap S^*$ . Assume that  $S$  is unramified over  $R$ . Then  $S^*$  is unramified over  $R^*$  and  $r(S^*: S) = r(R^*: R)$ ,  $\bar{r}(S^*: S) = \bar{r}(R^*: R)$ ,  $i(S^*: S) = i(R^*: R)$ , and  $g(S^*: S) \leq g(R^*: R)$ .*

*Proof.* Since the situation remains parallel if we pass to the completions of  $R, S, R^*, S^*$  [Section 2 of A2], we may assume that these local domains are complete to begin with. Let  $D, D^*, E, E^*$  be the residue fields of  $R, R^*, S, S^*$  respectively. Then our assumption implies that  $[E: D] = [E: D]_s = [L: K]$ . Since  $E/D$  is separable, there exists  $a$  in  $E$  such that  $E = D(a)$ . Fix an element  $A$  in  $S$  belonging to the residue class  $a$ . Let  $F(X)$  be the minimal monic polynomial of  $A$  over  $K$ . Then all the coefficients of  $F$  are in  $R$ . Let

$f(X)$  be the monic polynomial obtained by reducing the coefficients of  $F(X)$  modulo the maximal ideal in  $R$ . Let  $e$  denote the degree of  $F$  and hence also the degree of  $f$ . Now we must have  $f(a) = 0$  and hence  $e \geq [E:D]$ . However also  $e = [K(A):K] \leq [L:K]$ . Since  $[E:D] = [L:K]$ , we have  $e = [E:D] = [L:K]$  and hence  $K(A) = L$ . Consequently,  $K^*(A) = L^*$ . Let  $G(X)$  be the minimal monic polynomial  $A$  over  $K^*$ . Since  $R^*$  is normal,  $F = GH$  where  $G$  and  $H$  are monic polynomials with coefficients in  $R^*$ . Hence  $\Delta(G)$  divides  $\Delta(F)$  in  $R^*$ . Now  $\Delta(F)$  belongs to the residue class (modulo the maximal ideal in  $R$ )  $\Delta(f)$ . Since  $f$  is a separable polynomial,  $\Delta(f) \neq 0$  and hence  $\Delta(F)$  is a unit in  $R$  and hence a unit in  $R^*$ . Therefore  $\Delta(G)$  is a unit in  $R^*$  and consequently [see  $K$ ]  $S^*$  is unramified over  $R^*$ , i. e.,  $r(S^*: R^*) = \bar{r}(S^*: R^*) = i(S^*: R^*) = 1$ .

Now  $r(S^*: S)r(S: R) = r(S^*: R) = r(S^*: R^*)r(R^*: R)$  and hence  $r(S^*: S) = r(R^*: R)$ , and similarly  $\bar{r}(S^*: S) = \bar{r}(R^*: R)$  and  $i(S^*: S) = i(R^*: R)$ . Next,  $g(S^*: S)r(S^*: S) = [L^*: L]$  and  $g(R^*: R)r(R^*: R) = [K^*: K]$ ; since  $L^*$  is the compositum of  $K^*$  and  $L$ , any set of  $K$ -generators of the  $K$ -vector space  $K^*$  is also a set of  $L$ -generators of the  $L$ -vector space  $L^*$  and hence  $[L^*: L] \leq [K^*: K]$  and therefore in view of the equality  $r(S^*: S) = r(R^*: R)$  we can conclude that  $g(S^*: S) \leq g(R^*: R)$ .

*Remark 1.* The inequality  $g(S^*: S) \leq g(R^*: R)$  is in general false if we do not assume that  $S$  is unramified over  $R$ . This can be seen from the following example. Let  $k$  be an algebraically closed field of characteristic  $p$ , let  $u$  and  $x$  be independent variables over  $k$ , let  $n$  be an integer greater than 1 such that  $n$  is prime to  $p$  in case  $p \neq 0$ . Let  $K = k(u)((x))$ ,  $L^* = K(x^{1/n}, u^{1/n})$ ,  $K^* = K(x^{1/n})$ ,  $L = K(x^{1/n}u^{1/n})$ ,  $R = k(u)[[x]]$ . Then we must have

$$S^* = k(u^{1/n})[[x^{1/n}]], \quad R^* = k(u)[[x^{1/n}]], \quad S = k(u)[[(ux)^{1/n}]].$$

Consequently  $g(R^*: R) = 1$  and  $g(S^*: S) = n$ .

**LEMMA 2.** Let  $v$  be a real discrete valuation of a field  $K$ , let  $p$  be the characteristic of the residue field of  $v$ , let  $n_1, \dots, n_t$  be positive integers which are prime to  $p$  in case  $p \neq 0$ , let  $x_1, \dots, x_t$  be elements of  $K$  having  $v$ -value zero, and let  $L$  be a field generated over  $K$  by a certain number of roots of the polynomial  $\prod_{j=1}^t (X^{n_j} - x_j)$ . Then  $v$  is unramified in  $L$ . Now let  $K^*$  be a finite separable extension of  $K$ , let  $L^*$  be a composition of  $L$  and  $K^*$ , let  $w^*$  be an extension of  $v$  to  $L^*$ , let  $v^*$  be the restriction of  $w^*$  to  $K^*$  and let  $w$  be the restriction of  $w^*$  to  $L$ . Then  $r(w^*: w) = r(v^*: v)$ ,  $\bar{r}(w^*: w) = \bar{r}(v^*: v)$ ,  $i(w^*: w) = i(v^*: v)$ .

*Proof.*<sup>5</sup> If  $T$  is a finite separable extension of  $K$  and  $T^*$  is a finite separable extension of  $T$ , then  $v$  is unramified in  $T$  if and only if  $r(u:v) = 1$  for every  $T$ -extension  $u$  of  $v$ ; also if  $u$  is a  $T$ -extension of  $v$  and if  $u^*$  is a  $T^*$ -extension of  $u$  then  $r(u^*:v) = r(u^*:v)r(u:v)$ ,  $u(x_j) = 0$  for all  $j$ . Therefore it is clear that the first assertion will be proved if we show that  $v$  is unramified in  $K(z)$  where  $z$  is a root of  $f = f(X) = X^n - x_1$ ; let  $g(X)$  be the minimal polynomial of  $z$  over  $K$ , since  $R_v$  is integrally closed we must have  $f = gh$  where  $g$  and  $h$  are monic polynomials with coefficients in  $R_v$  and hence  $\Delta(g)$  divides  $\Delta(f)$  in  $R_v$ ; now  $\Delta(f)$  equals a power of  $n_1$  times a power of  $x_1$ ,  $x_1$  is a unit in  $R_v$  and  $n_1$  is not divisible by  $p$  if  $p \neq 0$  and hence  $n_1$  is also a unit in  $R_v$ , therefore  $\Delta(f)$  and consequently  $\Delta(g)$  is a unit in  $R_v$ , from this we conclude that  $v$  is unramified in  $K[z]$  (see [K]); this completes the proof of the first assertion. The second assertion now follows from Lemma 1.

LEMMA 3. Let  $v$  be a real discrete valuation of a field  $K$ , let  $p$  be the characteristic of the residue field of  $v$ , let  $x$  be a nonzero element of  $K$ , let  $q = v(x)$ , let  $n$  be a positive integer which is prime to  $p$  in case  $p \neq 0$ , let  $v$  be the greatest common divisor of  $n$  and  $q$ ,<sup>6</sup> let  $b = n/v$  and  $K^*$  be a field generated over  $K$  by one or more roots of the polynomial  $X^n - x$  over  $K$  and let  $v^*$  be an extension of  $v$  to  $K^*$ . Then  $r(v^*:v) = \bar{r}(v^*:v) = b$  and  $i(v^*:v) = 1$ .

*Proof.*<sup>5</sup> Fix  $y$  in  $K$  with  $v(y) = 1$  and let  $A = x/y^q$ . Then  $v(A) = 0$ , and hence  $\alpha \neq 0$ . Let  $L$  be a field generated over  $K$  by all the roots of the polynomial  $(X^n - A)(X^n - 1)$ , let  $L^*$  be a compositum of  $K^*$  and  $L$ , let  $w^*$  be an extension of  $v^*$  to  $L^*$  and let  $w$  be the  $L$ -restriction of  $w^*$ . Then by Lemma 2,  $r(v^*:v) = r(w^*:w)$ ,  $\bar{r}(v^*:v) = \bar{r}(w^*:w)$  and  $w(y) = 1$ . Therefore it is enough to show that  $r(w^*:w) = \bar{r}(w^*:w) = b$ . By our assumption,  $L^*$  contains a root  $z$  of  $X^n - x$  and  $L^*$  is generated over  $L$  by a certain number of roots of  $X^n - r$ . Let  $B$  be a root of  $X^n - A$  in  $L$  and let  $\xi = z/B$ . Then  $\xi \in L^*$  and  $\xi^n = y^q$ . Let  $t$  be any root of  $X^n - x$  (in some field extension of  $L^*$ ), then  $t^n = x = z^n$  and hence  $t/z$  is a root of  $X^n - 1$ ; therefore  $t/z \in L^*$  and hence  $t \in L^*$ . Therefore  $L^* = L(z) = L(\xi)$ . Let  $a = q/v$ . Now  $X^n - 1$  has all its roots in  $L$  and  $n$  is prime to  $p$  in case  $p \neq 0$  and hence  $n$  is prime to the characteristic of  $L$  in case this is different from zero, therefore  $L$  contains a primitive  $n$ -th root  $h$  of unity. Now  $(\xi^b/y^a)^v = \xi^{bv}/y^{av} = \xi^n/y^q = 1$

<sup>5</sup> Our assumption implies that all the extensions considered in the statement of the lemma and all the extensions to be considered in the proof are separable and hence we can use the methods of ramification and galois theories.

<sup>6</sup> Note that if  $q = 0$  then  $v = n$  and  $b = 1$ .

and  $b\nu = n$ , therefore  $\xi^b/y^a = h^{sb}$  where  $s$  is an integer. Let  $\eta = \xi h^{-s}$ . Then  $L^* = L(\eta)$  and  $\eta^b = y^a$ . If  $a = 0$  then  $b = 1$  and there is nothing to prove, so now assume that  $a \neq 0$ . Then  $a$  and  $b$  are coprime nonzero integers and hence we can find integers  $\alpha$  and  $\beta$  such that  $a\alpha + b\beta = 1$ . Let  $\xi = \eta^{\alpha y^\beta}$ . Then  $\xi^a = \eta^{\alpha a y^\beta} = \eta^{\alpha a} (y^a)^\beta = \eta^{\alpha a} (\eta^b)^\beta = \eta$  and hence  $L^* = L(\xi)$ . Also  $\xi^b = \eta^{\alpha b y^\beta} = (\eta^b)^{\alpha y^\beta} = (y^a)^{\alpha y^\beta} = y$ . Therefore  $\bar{r}(w^*: w) = w(y) \bar{r}(w^*: w) = w^*(y) = w^*(\xi^b) = b w^*(\xi)$  and hence  $\bar{r}(w^*: w) \geq b \geq [L^*: L]$ . Therefore  $\bar{r}(w^*: w) = b = [L^*: L]$ . Hence  $r(w^*: w) = \bar{r}(w^*: w) = b$ .

**PROPOSITION 1.** Let  $v$  be a real discrete valuation of a field  $K$ , let  $p$  be the characteristic of the residue field of  $v$ , let  $K^*$  be a finite separable extension of  $K$ , let  $v^*$  be an extension of  $v$  to  $K^*$ , let  $x, x_1, x_2, \dots, x_t$  ( $t \geq 0$ ) be nonzero elements of  $K$  such that  $v(x_j) = 0$  for  $j = 1, \dots, t$ ; and let  $n = n_0, n_1, \dots, n_t$  be positive integers which are not divisible by  $p$  in case  $p \neq 0$ . Let  $L$  be a field generated over  $K$  by one or more roots of the polynomial  $X^n - x$  and a certain number of roots of the polynomial  $\prod_{j=1}^t (X^{n_j} - x_j)$ , let  $L^*$  be a compositum of  $K^*$  and  $L$ , let  $w^*$  be an extension of  $v^*$  to  $L^*$  and let  $w$  be the restriction of  $w^*$  to  $L$ . Let  $q = v(x)$  and let  $a = \bar{r}(v^*: v)$ . Let  $v$  be the greatest common divisor of  $q$  and  $n$ .<sup>6</sup> Let  $b = n/v$ . Let  $d$  be the greatest common divisor of  $a$  and  $b$ . Let  $e = a/d$ . Then (i)  $\bar{r}(w^*: w) = e$  and  $i(w^*: w) = i(v^*: v)$ . Furthermore (ii) if  $q = 1$  and  $a$  divides  $n$  then  $\bar{r}(w^*: w) = 1$ ; (iii) if  $r(v^*: v) \not\equiv 0 \pmod{p}$  in case  $p \neq 0$  and always in case  $p = 0$  we can choose  $x$  and  $n$  such that  $q = 1$  and  $a$  divides  $n$ , and then  $w^*$  is unramified over  $w$ ; and (iv) if  $q = 1$ , if  $a$  divides  $n$ , if  $i(v^*: v) = 1$ , if  $T$  is a finite separable extension of  $L$ ; if  $T^*$  is the compositum of  $T$  and  $K^*$ , if  $u^*$  is a  $T^*$ -extension of  $v^*$  and if  $u$  is the  $T$ -restriction of  $u^*$ , then  $u^*$  is unramified over  $u$ .

*Proof.*<sup>5</sup> (ii) and (iii) follow from (i), and (iv) follows from (iii) in view of Lemma 1; hence it is enough to prove (i).

Let  $K_1$  be a field generated over  $K$  by all the roots of the polynomial  $\prod_{j=1}^t (X^{n_j} - x_j)$ ; let  $K_1^*, L_1, L_1^*$  be the compositums of  $K_1$  respectively with  $K^*, L, L^*$ ; let  $w_1^*$  be an extension of  $w^*$  to  $L_1^*$  and let  $v_1, v_1^*, w_1$  be the restrictions of  $w_1^*$  respectively to  $K_1, K_1^*, L_1$ . Then by Lemma 2 we have that

$$\begin{aligned} \bar{r}(v_1^*: v_1) &= \bar{r}(v^*: v) = a, & i(v_1^*: v_1) &= i(v^*: v), \bar{r}(w_1^*: w) = \\ \bar{r}(w^*: w), i(w_1^*: w_1) &= i(w^*: w); \text{ and } v_1(x) = v(x) = q \cdots (I) \end{aligned}$$

Now  $n = b\nu$  and  $q = c\nu$  where  $b$  and  $c$  are coprime nonzero integers and

$a = ed$  and  $b = fd$  where  $e$  and  $f$  are coprime nonzero integers. Applying Lemma 3 to  $L_1/K_1$  we have that

$$\bar{r}(w_1: v_1) = b \text{ and } i(w_1: v_1) = 1 \cdot \cdot \cdot \text{ (II)}$$

Now  $aq = (ec)(dv)$  and  $n = (f)(dv)$ ; since  $b$  and  $c$  are coprime,  $f$  and  $c$  must also be coprime; also  $f$  and  $e$  are coprime and hence  $f$  and  $ec$  are coprime; therefore  $dv$  is the greatest common divisor of  $aq$  and  $n$ , and  $n = (f)(dv)$ . Also by (I),  $v^*_1(x) = \bar{r}(v^*_1: v_1)v_1(x) = aq$ . Hence applying Lemma 3 to  $L^*_1/K^*_1$  we get

$$\bar{r}(w^*_1: v^*_1) = f \text{ and } i(w^*_1: v^*_1) = 1 \cdot \cdot \cdot \text{ (III)}$$

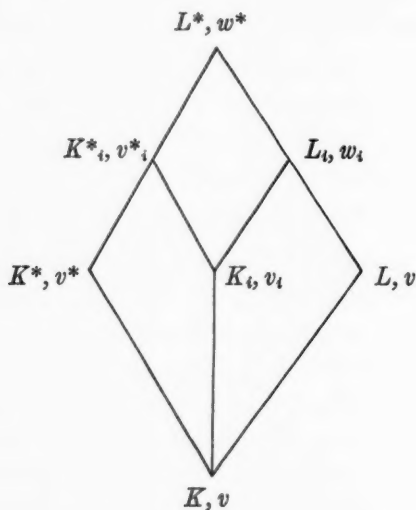
Now  $\bar{r}(w^*_1: v^*_1)\bar{r}(v^*_1: v_1) = \bar{r}(w^*_1: v_1) = \bar{r}(w^*_1: w_1)\bar{r}(w_1: v_1)$  and hence by (I, II, III) we get  $fa = \bar{r}(w^*_1: w_1)b$  and substituting  $ed$  for  $a$  and  $fd$  for  $b$  this gives us  $\bar{r}(w^*_1: w_1) = e$  and hence by (I):  $\bar{r}(w^*: w) = e$ . Again  $i(w^*_1: v^*_1)i(v^*_1: v_1) = i(w^*_1: v_1) = i(w^*_1: w_1)i(w_1: v_1)$  and hence by (I, II, III) we have:  $i(w^*: w) = i(v^*: v)$ .

**PROPOSITION 2.** *Let  $K$  be an  $n$  dimensional algebraic function field over an algebraically closed ground field  $k$ , let  $V$  be a nonsingular projective model of  $K/k$ , let  $W$  be a pure  $n-1$  dimensional subvariety of  $K$  and let  $W_1, \cdot \cdot \cdot, W_t$  be the irreducible components of  $W$ . Assume that the following conditions are satisfied: (i)  $\pi'(V - W_2 - W_3 - \cdot \cdot \cdot - W_t)$  is abelian; (ii)  $Q(W_1, V)$  does not split (i.e. there is a unique local ring lying above it) in any finite separable extension of  $K$  which is tamely ramified over  $V$  and for which the branch locus over  $V$  is contained in  $W$ ; (iii) some nonzero integral multiple of  $W_1$  is linearly equivalent to some integral multiple of  $W_2$ . Then  $\pi'(V - W)$  is abelian.*

*Proof.*<sup>7</sup> We have to show that if  $K^*/K$  is a galois extension such that  $K^*/V$  is tamely ramified and  $\Delta(K^*/V) \subset W$  then  $G(K^*/K)$  is abelian. Assumption (iii) means that there exists  $y$  in  $K$  such that  $(y) = \alpha W_1 - \beta W_2$  where  $\alpha$  and  $\beta$  are nonzero integers. Let  $v$  be the real discrete valuation of  $K$  having  $Q(W_1, V)$  as its valuation ring and let  $v^*$  be the unique  $K^*$ -extension of  $v$ . Let  $a = \bar{r}(v^*: v) = r(v^*: v)$ . Let  $\alpha_1 = 1$  in case  $p = 0$  and  $\alpha_1$  = the highest power of  $p$  which divides  $\alpha$  in case  $p \neq 0$ . Let  $\alpha_2 = \alpha/\alpha_1$  and let  $m = a\alpha_2$ . Then  $m$  is prime to  $p$  in case  $p \neq 0$ . Let  $L$  be the field generated over  $K$  by a root  $x$  of the polynomial  $X^m - y$ . Since  $k$  is algebraically closed,  $L/K$  is a galois extension,  $G(L/K)$  is cyclic and its order is prime to  $p$  in case  $p \neq 0$  and hence  $L/V$  is tamely ramified. Since  $(y) = \alpha W_1 - \beta W_2$ ,

<sup>7</sup> All extensions of  $K$  are to be taken in some fixed algebraic closure of  $K$ .

Lemma 2 tells us that  $\Delta(L/V) \subset W_1 \cup W_2 \subset W$ . Let  $L^*$  be the compositum of  $L$  and  $K^*$ . Then  $L^*/K$  is galois and by Lemma 12 of Part I,  $L^*/V$  is tamely ramified and  $\Delta(L^*/V) \subset W$ . Hence by assumption (ii), there is only one  $L^*$ -extension  $w^*$  of  $v$  and it is clear that  $v^*$  is the  $K^*$ -restriction of  $w^*$ . Let  $K_i$  be the inertia field of  $w^*/v$ , let  $K_i^*$  and  $L_i$  be the compositums of  $K_i$  with  $K^*$  and  $L$  respectively, let  $v_i, v_i^*, w_i$  be the restrictions of  $w^*$  to  $K_i, K_i^*, L_i$  respectively. Then by Lemmas 1, 2, 13 and 17 of Part I we get (1)  $K_i/K$  is galois, (2)  $g(w^*: v_i) = 1$ , (3)  $r(v_i: v) = 1$ , (4)  $\Delta(K_i/K) \subset W_2 \cup \dots \cup W_t$ .



Also since  $L^*/V$  is tamely ramified, so is  $K_i/V$  and hence in view of (4) assumption (i) tells us that  $G(K_i/K)$  is abelian, and since  $L/K$  is cyclic we conclude that (5)  $L_i/K$  is galois and  $G(L_i/K)$  is abelian. From (3) and Lemma I we get (6)  $\bar{r}(v_i^*: v_i) = a$ . Now it is clear that (7)  $L^*$  is the compositum of  $K_i^*$  and  $L_i$ ,  $L_i = K_i(x)$ , and  $x^m = y$ . Also  $(y) = \alpha W_1 - \beta W_2$  implies that  $v(y) = \alpha$  and hence by (3) we get (8)  $v_i(y) = \alpha = \alpha_1 \alpha_2$ . Let  $\mu$  be the greatest common divisor of  $m$  and  $\alpha$ , and let  $b = m/\mu$ . Now  $a$  is not divisible by  $p$  in case  $p \neq 0$  and hence always  $a$  and  $\alpha_1$  are coprime; since  $m = a\alpha_2$  we can conclude that  $\mu = \alpha_2$  and hence  $b = a$ . In view of (6, 7, 8), Proposition 1(i) applied to the top quadrilateral in the diagram now yields (9)  $r(w^*: w_i) = 1$ . Also (2) or alternatively [A4, Proposition 1.49] tell us that (10)  $g(w^*: w_i) = 1$ . Since  $w^*$  is the only  $L^*$ -extension of  $v_i$ , (9) and (10) now yield that  $L^* = L_i$  and hence by (5),  $G(L^*/K)$  is abelian. Therefore  $G(K^*/K)$  is abelian.

**3. Main result.** Throughout this section  $K$  will denote an  $n$ -dimensional algebraic function field over an algebraically closed ground field  $k$  of characteristic  $p$ ,  $V$  will denote a simply connected nonsingular projective model of  $K/k$ ,  $W$  will denote a pure  $(n-1)$ -dimensional subvariety of  $V$  and  $W_1, \dots, W_t$  will denote the irreducible components of  $W$ .

**PROPOSITION 3.** *Assume that for any  $j$  and  $k$  some nonzero integral multiple of  $W_j$  is linearly equivalent to some integral multiple of  $W_k$ , and that the components  $W_j$  can be labelled such that for  $j=1, \dots, t$ ,  $Q(W_j, V)$  does not split in any finite separable extension  $K^*$  of  $K$  for which  $K^*/V$  is tamely ramified and  $\Delta(K^*/V) \subset W_j \cup W_{j+1} \cup \dots \cup W_t$ . Then  $\pi'(V-W)$  is abelian and  $\pi^*(V-W) = \pi'(V-W)$ .*

*Proof.* That  $\pi'(V-W)$  is abelian follows for  $t=1$  as in the proof of Theorem 4 of Section 14 of Part I and from this the general case follows by induction in view of Proposition 2 above. Now assume that  $p \neq 0$  and let  $K^*$  be a galois extension of  $K$  such that  $K^*/V$  is tamely ramified and  $\Delta(K^*/V) \subset W$ . Then  $K^*/V$  is abelian and hence for each  $j$ , the inertia groups over  $K$  of any local ring in  $K^*$  lying above  $Q(W_j, V)$  coincide with each other, let  $G_j$  be this common group, let  $G$  be the subgroup of  $G(K^*/K)$  generated by  $G_1, \dots, G_t$  and let  $L$  be the fixed field of  $G$ . Then by Lemmas 13 and 17 of Part I,  $L/V$  is unramified and hence  $L=K$ . Now the order of each subgroup  $G_j$  is prime to  $p$  and hence the order of  $G(K^*/K)$  is also prime to  $p$ . Therefore  $\pi^*(V-W) = \pi'(V-W)$ .

**THEOREM 1.** *Assume that for each  $j$  and  $k$  some nonzero integral multiple of  $W_j$  is linearly equivalent to some integral multiple of  $W_k$ , and either that  $W$  has only strong normal crossings and  $\dim |W_j| > 1$  for each  $j$  or that  $n=2$  and for some labelling of the components  $W_j$  we have  $\dim |W_j| > 1 + v(W_j, W_j \cup W_{j+1} \cup \dots \cup W_t; V)$  for each  $j$ . Then  $\pi'(V-W)$  is abelian and is generated by  $t$ -generators.*

*Proof.* In view of Proposition 3, that  $\pi'(V-W)$  is abelian follows from Proposition 6 of Section 11 of Part I and Proposition 14 of Section 9 of Part II respectively. That  $\pi'(V-W)$  is generated by  $t$  generators has been proved in Theorem 2 of Part I<sup>2</sup> and Theorem 2 of Part II.

**4. Applications.** As in the proofs of Theorem 3 of Section 13 of Part I<sup>2</sup> and Theorem 3 of Section 10 of Part II, Theorem 1 of the previous section now gives the following theorem which subsumes these results of Parts I and II. This is the theorem of which we spoke of in Remark 9 of Section 10 of Part II.

**THEOREM 2.** Let  $P_n$  be the  $n$  dimensional projective space ( $n > 1$ ) over an algebraically closed ground field  $k$  of characteristic  $p$ , let  $W$  be a hypersurface in  $P_n$  with irreducible components  $W_1, \dots, W_t$ , let  $g^*_j$  be the degree of  $W_j$ ; let  $d = 1$  in case  $p = 0$  and  $d =$  the highest power of  $p$  which divides  $g^*_1, \dots, g^*_t$  in case  $p \neq 0$ ; let  $g_j = g^*_j d^{-1}$ , and let  $G$  be the abelian group generated by  $t$  generators  $a_1, \dots, a_t$  with the only relation

$$a_1^{g_1} \cdots a_t^{g_t} = 1.$$

Assume either that  $W$  has only strong normal crossings or that  $n = 2$  and the components  $W_j$  can be labelled so that

$$\frac{1}{2}g^*_j(g^*_j + 3) > 1 + v(W_j, W_j \cup W_{j+1} \cup \cdots \cup W_t; P_2).$$

Then  $G$  is a tame fundamental parent group of  $V - W$ . Also  $\pi^*(V - W) = \pi'(V - W)$  and hence  $G$  is a reduced fundamental parent group of  $V - W$  as well.  $G$  is a direct product of a free abelian group on  $t - 1$  generators and a cyclic group of order equal to the greatest common divisor of  $g_1, \dots, g_t$ ; i.e. equal to the greatest common divisor of  $g^*_1, \dots, g^*_t$  in case  $p = 0$  and to the part of this prime to  $p$  in case  $p \neq 0$ .

**Remark 2.** Results similar to Proposition 3 with regard to Theorem 2 of Section 12 of Part I, Theorems 1, 2 of Section 9 of Part II and Theorem 2 of this section clearly hold. Also all the remarks made in Parts I and II concerning Theorems 2 and 3 of Part I and Theorems 1, 2 and Part II now are valid in the situations of Theorems 1 and 2 of this paper. Finally it is clear that in the classical case we now have the form of Proposition 19 of Section 10 of Part II corresponding to Theorems 1 and 2 above.

**Remark 3.** Let the notation be as in Definition 7 of Section 6 of Part II and assume that  $A = \text{Rad}_R A$ . Then instead of treating a 2-fold normal crossing in a way similar to an  $s$ -fold ordinary point ( $s \geq 2$ ) one might conceivably have thought of defining " $v(A, A; R, R) = 0$  if  $A$  has a normal crossing at  $R$ ." This would have been inappropriate, for otherwise the above Theorem 2 would have been false. To show this let us further consider the example given in Remark 5 of Section 8 of Part II. We shall use the notation of that Remark. Let  $v$  be the real discrete valuation of  $K/k$  having center  $W_1$  on  $V$ . Then as shown,  $v$  splits into two valuations in  $K^*$ , let these be  $v'$  and  $v^*$ . Let  $a = yz - x$  and  $b = yz + x$ . Then  $v(ab) = v(f) = 1$ , and  $a$  and  $b$  generate distinct prime ideals in  $R_1$  and hence after a suitable labelling of  $v', v^*$  we must have  $v'(a) = 1, v'(b) = 0, v^*(a) = 0, v^*(b) = 1$ . Let  $W'$  and  $W^*$  be the irreducible components of  $\phi^{-1}(W_1)$  corresponding to  $v',$  and  $v^*$

respectively. Now the  $K^*/K$  norm of  $a$  is  $f$  and hence the part of the divisor of  $a$  on  $V^*$  at finite distance equals  $W'$ .<sup>8</sup> Let  $n$  be any integer greater than 1 such that  $n$  is prime to  $p$  in case  $p \neq 0$ . Let  $K_1 = K^*(a^{1/n})$ . Then  $K_1/V$  is tamely ramified and  $\Delta(K_1/V) \subset W$ . However  $v'$  is ramified in  $K_1$  while  $v^*$  is not and consequently  $K_1/K$  is not galois and hence  $\pi'(V-W)$  cannot be abelian, i.e., Theorem 2 does not apply to  $V-W$ .

Next, one can easily verify the following: (1) the only singularity of  $W_1$  is the 2-fold normal crossing at  $P: X=Y=0$  and neither  $L$  nor  $L_\infty$  pass through this; (2) at  $P': (X=0, Y=1)$ ,  $L$  and  $W_1$  have a 2-fold contact and  $L_\infty$  does not go through  $P'$ ; (3) at the point  $P^*$  at infinity in the direction  $Y=0$ ,  $W_1$  and  $L_\infty$  have a 3-fold contact and  $L$  has a normal crossing with  $W_1$ . Therefore by the results of Section 7 of Part II we get  $\nu(W_1, W; P, V) = 3$ ,  $\nu(W_1, W; P', V) = 2$  and  $\nu(W_1, W; P^*, V) = 3$ . Hence  $\nu(W_1, W; V) = 8$ . Also  $\dim |W_1| = 9$  and  $9 \not\geq 1 + 8$ . This accounts for the nonapplicability of Theorem 2. However had we set  $\nu(W_1, W; P, V) = 0$  at the 2-fold normal crossing  $P$  then  $\nu(W_1, W; V)$  would have been equal to 5 and we would have had  $9 \geq 1 + 5$ .

*Remark 4.* Let  $P^2$  be a projective plane over an algebraically closed ground field  $k$  of characteristic  $p$ , let  $W$  be a curve on  $P^2$ , and let  $W_1, \dots, W_t$  be the irreducible components of  $W$ . A weaker form of Proposition 3 is this: (A) If for  $j=1, \dots, t$ ,  $Q(W_j, P^2)$  does not split in any member of  $\Omega'(P^2 - W)$ , then  $\pi'(P^2 - W)$  is abelian. The corresponding form of Theorem 1 is this: (B) If  $\dim W_j > 1 + \nu(W_j, W; P^2)$  for  $j=1, \dots, t$ , then  $\pi'(P^2 - W)$  is abelian. Here we shall show that the converse of (A) does not hold, i.e., some  $W_j$  can split in some member of  $\Omega'(P^2 - W)$  and  $\pi'(P^2 - W)$  may still be abelian. The same examples will also show that there exists a tamely ramified covering  $f: V \rightarrow P^2$ , and two distinct lines  $L_1, L_2$  such that  $\pi'(V - L^* - f^{-1}(L_2)) = 1$  where  $L^*$  is an irreducible component of  $f^{-1}(L_1)$ .

Let  $P^3$  be a projective three space over  $k$  with affine coordinates  $X, Y, Z$ ; let  $V$  be a nonsingular surface in  $P^3$ ; consider  $P^2$  to be the  $(X, Y)$ -plane; and let  $f$  be the projection of  $V$  onto  $P^2$  along the  $Z$ -axis. *Example 1.* ( $p \neq 2$ ).  $t=3$ ;  $V: Z^2 + XY - 1 = 0$ ;  $W_1: XY - 1 = 0$ ;  $W_2: X = 0$ ;  $W_3 =$  the line at infinity. Then  $\Delta(V/P^2) = W_1 \subset W$ .  $Z^2 + XY - 1 \equiv (Z+1)(Z-1) \pmod{X}$  and hence  $f^{-1}(W_2)$  has two irreducible components  $W^*$  and  $W'$ . Now  $\nu(W_1, W; P^2) = 2$  and  $\dim |W_1| = 5$ . Hence by Theorem 1,  $\pi'(P^2 - W)$  is abelian. Suppose if possible that  $\pi'(V - W^* - f^{-1}(W_3)) \neq 1$ . Then there exists a finite separable extension  $K$  of  $k(V)$  other than  $k(V)$  such that  $K/V$

<sup>8</sup> One can also easily show that  $\phi^{-1}(L_\infty)$  is irreducible and  $(a) = W_1 - \phi^{-1}(L_\infty)$ .

is tamely ramified and  $\Delta(K/V) \subset W^* \cup f^{-2}(W_3)$ . Then  $K/P^2$  is tamely ramified and  $\Delta(K/P^2) \subset W$ . Since  $\pi'(P^2 - W)$  is abelian,  $K/P^2$  is galois and since  $W^*$  and  $W'$  are components of  $f^{-1}(W_2)$ , we must have  $\Delta(K/V) \subset f^{-1}(W_3)$  and hence  $\Delta(K/P^2) \subset W_1 \cup W_3$ . From this, in view of [Part I, Section 13] we conclude that  $f^{-1}(W_1)$  must be ramified in  $K$ . This is a contradiction. Therefore  $\pi'(W - W^* - f^{-1}(W_3)) = 1$ . If  $k$  is the field of complex numbers then  $\pi_1(V - W^* - f^{-1}(W_3)) = 1$ . This can be seen by noting that  $V - W^* - f^{-1}(W_3)$  is biregularly equivalent to an immediate quadratic transform of a complex affine plane.

*Example 2.* ( $p \neq 3$ ).  $t = 3$ ;  $V: Z^3 - Y^3 - X(X+a)(X+b) = 0$  where  $a$  and  $b$  are distinct nonzero elements of  $k$ ;  $W_1: Y^3 + X(X+a)(X+b) = 0$ ;  $W_2: X = 0$ ;  $W_3 = \text{line at infinity}$ . Then  $v(W_1, W; P^2) = 3$ ,  $\dim |W_1| = 9$ , and

$$Z^3 - Y^3 - X(X+a)(X+b) \equiv (Z - hY)(Z - h^2Y)(Z - Y) \pmod{X}$$

where  $h$  is a primitive cube root of 1, etc.

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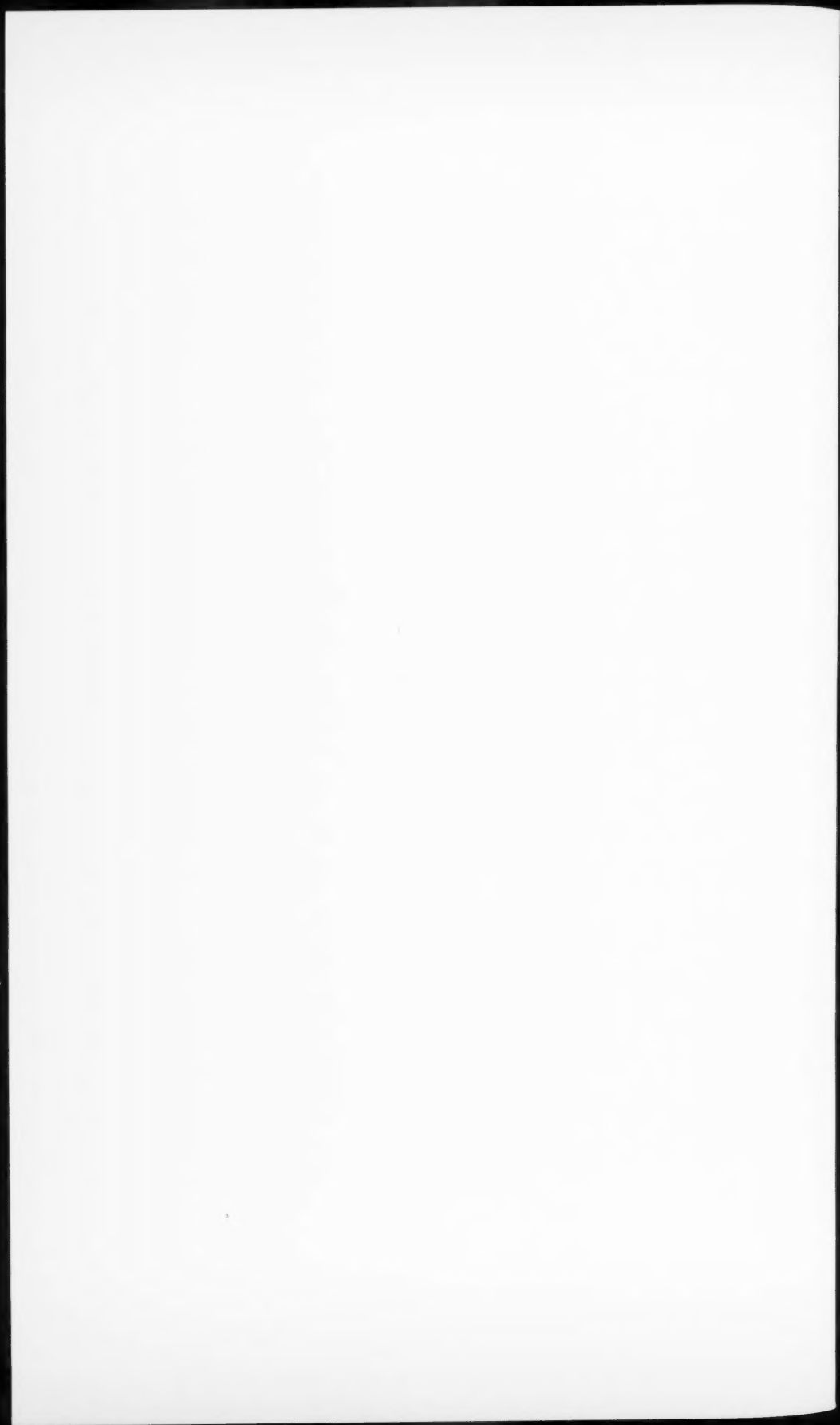
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